I. Development of the Virial Theorem

1. The Basic Equations of Structure

Before turning to the derivation of the virial theorem, it is appropriate to review the origin of the fundamental structural equations of stellar astrophysics. This not only provides insight into the basic conservation laws implicitly assumed in the description of physical systems, but by their generality and completeness graphically illustrates the complexity of the complete description that we seek to circumvent. Since lengthy and excellent texts already exist on this subject, our review will of necessity be a sketch. Any description of a physical system begins either implicitly or explicitly from certain general conservation principles. Such a system is considered to be a collection of articles, each endowed with a spatial location and momentum which move under the influence of known forces. If one regards the characteristics of spatial position and momentum as being highly independent, then one can construct a multi-dimensional space through which the particles will trace out unique paths describing their history.

This is essentially a statement of determinism, and in classical terms is formulated in a six-dimensional space called phase-space consisting of three spatial dimensions and three linearly independent momentum dimensions. If one considers an infinitesimal volume of this space, he may formulate a very general conservation law which simply says that the divergence of the flow of particles in that volume is equal to the number created or destroyed within that volume.

The mathematical formulation of this concept is usually called the Boltzmann transport equation and takes this form:

$$\frac{\partial \psi}{\partial t} + \sum_{i=1}^{3} \dot{x}_i \frac{\partial \psi}{\partial x_i} + \sum_{i=1}^{3} \dot{p}_i \frac{\partial \psi}{\partial p_i} = S \quad ,$$

or in vector notation 1.1.1

$$\frac{\partial \psi}{\partial t} + \mathbf{v} \bullet \nabla \psi + \mathbf{f} \bullet \nabla_p \psi = S \quad ,$$

where ψ is the density of points in phase space, **f** is the vector sum of the forces acting on the particles and S is the 'creation rate' of particles within the volume. The homogeneous form of this equation is often called the Louisville Theorem and would be discussed in detail in any good book on Classical Mechanics.

A determination of ψ as a function of the coordinates and time constitutes a complete description of the system. However, rarely is an attempt made to solve equation (1.1.1) but rather simplifications are made from which come the basic equations of stellar structure. This is generally done by taking 'moments' of the equations with respect to the various coordinates. For example, noting that the integral of ψ over all velocity space yields the matter density ρ and that no particles can exist with unbounded momentum, averaging equation (1.1.1) over all velocity space yields

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\mathbf{u}\rho) = \overline{S} \quad , \qquad 1.1.2$$

where **u** is the average stream velocity of the particles and is defined by

$$\mathbf{u} = \frac{1}{\rho}\int \psi \mathbf{v} dv \quad . \qquad 1.1.3$$

For systems where mass is neither created nor destroyed $\overline{S} = 0$, and equation (1.1.2) is just a statement of the conservation of mass. If one multiplies equation (1.1.1) by the particle velocities and averages again over all velocity space he will obtain after a great deal of algebra the Euler-Lagrange equations of hydrodynamic flow

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \bullet \nabla)\mathbf{u} = -\nabla \Psi - \frac{1}{\rho}\nabla \bullet \mathfrak{P} - \frac{1}{\rho}\int \mathbf{S}(\mathbf{v} - \mathbf{u}) dv \quad . \qquad 1.1.4$$

Here the forces **f** have been assumed to be derivable from a potential Ψ. The symbol $\mathfrak{P}$ is known as the pressure tensor and has the form

$$\mathfrak{P} = \int \psi (\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u}) dv \, . \qquad 1.1.5$$

These rather formidable equations simplify considerably in the case where many collisions randomize the particle motion with respect to the mean stream velocity **u** .Under these conditions the last term on the right of equation (1.1.4) vanishes and the pressure tensor becomes diagonal with each element equal. Its divergence then becomes the gradient of the familiar scalar known as the gas pressure P. If we further consider only systems exhibiting no stream motion we arrive at the familiar equation of hydrostatic equilibrium

$$\nabla P = -\rho \nabla \Psi \quad . \qquad 1.1.6$$

Multiplying equation (1.1.1) by **v** and averaging over **v**, has essentially turned the Boltzmann transport equation into an equation expressing the conservation of momentum. Equation (1.1.6) along with Poisson's equation for the sources of the potential

$$\nabla^2 \Psi = -4\pi G\rho \, , \qquad 1.1.7$$

constitute a complete statement of the conservation of momentum.

Multiplying equation (1.1.1) by $\mathbf{v} \bullet \mathbf{v}$ or v^2 and averaging over all velocity space will produce an equation which represents the conservation of energy, which when combined with the ideal gas law is

$$\rho \frac{dE}{dt} + \rho \nabla \bullet \mathbf{v} = \rho\varepsilon + \chi - \nabla \bullet \mathbf{F}, \qquad 1.1.8$$

where **F** is the radiant flux, ε the total rate of energy generation and χ is the energy generated by viscous motions. If one has a machine wherein no mass motions exist and all energy flows by radiation, we have a statement of radiative equilibrium;

$$\nabla \bullet \mathbf{F} = \rho\varepsilon \quad . \qquad 1.1.9$$

For static configurations exhibiting spherical symmetry these conservative laws take their most familiar form:

Conservation of mass $\quad \dfrac{dm(r)}{dr} = 4\pi r^2 \rho$.

Conservation of momentum $\quad \dfrac{dP(r)}{dr} = -\dfrac{Gm(r)\rho}{r^2}$. $\qquad 1.1.10$

Conservation of energy $\quad \dfrac{dL(r)}{dr} = 4\pi r^2 \rho\varepsilon, \qquad L(r) = 4\pi r^2 F$.

2. The Classical Derivation of the Virial Theorem

The virial theorem is often stated in slightly different forms having slightly different interpretations. In general, we shall repeat the version given by Claussius and express the virial theorem as a relation between the average value of the kinetic and potential energies of a system in a steady state or a quasi-steady state. Since the understanding of any theorem is related to its origins, we shall spend some time deriving the virial theorem from first principles. Many derivations of varying degree of completeness exist in the literature. Most texts on stellar or classical dynamics (e.g. Kurth[1]) derive the theorem from the Lagrange identity. Landau and Lifshitz[2] give an eloquent derivation appropriate for the electromagnetic field which we shall consider in more detail in the next section. Chandrasekhar[3] follows closely the approach of Claussius while Goldstein[4] gives a very readable vector derivation firmly rooted in the original approach and it is basically this form we shall develop first. Consider a general system of mass points m_i with position vectors $\mathbf{r}_i$ which are subjected to applied forces (including any forces of constraint) $\mathbf{f}_i$. The Newtonian equations of motions for the system are then

$$\dot{\mathbf{p}}_i = \frac{d(m_i \mathbf{v}_i)}{dt} = \mathbf{f}_i \quad . \qquad 1.2.1$$

Now define $$G = \sum_i \mathbf{p}_i \bullet \mathbf{r}_i = \sum_i m_i \frac{d\mathbf{r}_i}{dt} \bullet \mathbf{r}_i = \tfrac{1}{2}\sum_i m_i \frac{d(\mathbf{r}_i \bullet \mathbf{r}_i)}{dt} = \tfrac{1}{2}\frac{d}{dt}\left(\sum_i m_i r_i^2\right) . \qquad 1.2.2$$

The term in the large brackets is the moment of inertia (by definition) about a point and that point is the *origin* of the coordinate system for the position vectors $\mathbf{r}_i$. Thus, we have

$$G = \tfrac{1}{2}\frac{dI}{dt} \quad , \qquad 1.2.3$$

where I is the moment of inertia about the *origin* of the coordinate system.

Now consider

$$\frac{dG}{dt} = \sum \dot{\mathbf{r}}_i \bullet \mathbf{p}_i + \sum_i \dot{\mathbf{p}}_i \bullet \mathbf{r}_i \quad , \qquad 1.2.4$$

but $$\sum_i \dot{\mathbf{r}}_i \bullet \mathbf{p}_i = \sum_i m_i \dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i = \sum_i m_i v_i^2 = 2T \ , \qquad 1.2.5$$

where T is the total kinetic energy of the system with respect to the origin of the coordinate system. However, since $\dot{\mathbf{p}}_i$ is really the applied force acting on the system (see equation 1.2.1), we may rewrite equation (1.2.4) as follows:

$$\frac{dG}{dt} = 2T + \sum_i \mathbf{f}_i \bullet \mathbf{r}_i \quad . \qquad 1.2.6$$

The last term on the right is known as the Virial of Claussius. Now consider the Virial of Claussius. Let us assume that the forces $\mathbf{f}_i$ obey a power law with respect to distance and are derivable from a potential. The total force on the ith particle may be determined by summing all the forces acting on that particle. Thus

$$\mathbf{f}_i = \sum_{j \neq i} \mathbf{F}_{ij} \quad , \qquad 1.2.7$$

where $\mathbf{F}_{ij}$ is the force between the ith and jth particle. Now, if the forces obey a power law and are derivable from a potential then,

$$\mathbf{F}_{ij} = \nabla_i m_i \Phi(r_{ij}) = -\nabla_i a_{ij} r_{ij}^n \quad . \qquad 1.2.8$$

The subscript on the ∇-operator implies that the gradient is to be taken in a coordinate system having the ith particle at the origin. Carrying out the operation implied by equation (1.2.8), we have

$$\mathbf{F}_{ij} = -n\, a_{ij} r_{ij}^{(n-2)} (\mathbf{r}_i - \mathbf{r}_j) \quad . \qquad 1.2.9$$

Now since the force acting on the ith particle due to the jth particle may be paired off with a force exactly equal and oppositely directed, acting on the jth particle due to the ith particle, we can rewrite equation (1.2.7) as follows:

$$\mathbf{f}_i = \sum_j \mathbf{F}_{ij} = \sum_{j>i} \mathbf{F}_{ij} + \mathbf{F}_{ji} \quad . \tag{1.2.10}$$

Substituting equation (1.2.10) into the definition of the Virial of Claussius, we have

$$\sum_i \mathbf{f}_i \bullet \mathbf{r}_i = \sum_i \sum_{j>i} \mathbf{F}_{ij} \bullet \mathbf{r}_i + \mathbf{F}_{ji} \bullet \mathbf{r}_j \quad . \tag{1.2.11}$$

It is important here to notice that the position vector $\mathbf{r}_i$, which is 'dotted' into the force vector, bears the same subscript as the first subscript on the force vector. That is, the position vector is the vector from the origin of the coordinate system to the particle being action upon. Substitution of equation (1.2.9) into equation (1.2.11) and then into equation (1.2.6) yields:

$$\frac{dG}{dt} = 2T - n\mathcal{U} \quad , \tag{1.2.12}$$

where $\mathcal{U}$ is the total potential energy.[1.1] For the gravitational potential n = -1, and we arrive at a statement of what is known as Lagranges' Identity:

$$\frac{dG}{dt} = \tfrac{1}{2}\frac{d^2I}{dt^2} = 2T + \Omega \quad . \tag{1.2.13}$$

To arrive at the usual statement of the virial theorem we must average over an interval of time (T_0). It is in this sense that the virial theorem is sometimes referred to as a statistical theorem. Therefore, integrating equation (1.2.12), we have

$$\frac{1}{T_0}\int_0^{T_0} \frac{dG}{dt} dt = \frac{2}{T_0}\int_0^{T_0} T(t)dt \quad - \quad \frac{n}{T_0}\int_0^{T_0} \mathcal{U}(t)dt \quad . \tag{1.2.14}$$

and, using the definition of average value we obtain:

$$\frac{1}{T_0}\left[G(T_0) - G(0)\right] = 2\overline{T} - n\overline{\mathcal{U}} \quad . \tag{1.2.15}$$

If the motion of the system over a time T_0 is periodic, then the left-hand side of equation (1.2.15) will vanish. Indeed, if the motion of the system is bounded [i.e., $G(t) < \infty$], then we may make the left hand side of equation (1.2.15) as small as we wish by averaging over a longer time. Thus, if a system is in a steady state the moment of inertia (I) is constant and for systems governed by gravity

$$2\overline{T} + \overline{\Omega} = 0 \quad . \tag{1.2.16}$$

It should be noted that this formulation of the virial theorem involves time averages of indeterminate length. If one is to use the virial theorem to determine whether a system is in accelerative expansion or contraction, then he must be very careful about how he obtains the average value of the kinetic and potential energies.

3. Velocity Dependent Forces and the Virial Theorem

There is an additional feature of the virial theorem as stated in equation (1.2.16) that should be mentioned. If the forces acting on the system include velocity dependent forces, the result of the virial theorem is unchanged. In order to demonstrate this, consider the same system of mass points m_i subjected to forces $\mathbf{f}_i$ which may be divided into velocity dependent $(\overline{\mathbf{w}}_i)$ and velocity independent forces ($\mathbf{z}_i$). The equations of motion may be written as:

$$\dot{\mathbf{p}}_i = \mathbf{f}_i = \overline{\mathbf{w}}_i + \mathbf{z}_i \quad . \tag{1.3.1}$$

Substituting into equation (1.2.6), we have

$$\frac{dG}{dt} - \sum_i \overline{\mathbf{w}}_i \bullet \mathbf{r}_i = 2T + \sum_i \mathbf{z}_i \bullet \mathbf{r}_i \quad . \tag{1.3.2}$$

Remembering that the velocity dependent forces may be rewritten as

$$\overline{\mathbf{w}}_i = \alpha_i \mathbf{v}_i = \alpha_i \frac{d\mathbf{r}_i}{dt} \quad . \tag{1.3.3}$$

We may again average over time as in equation (1.2.12). Thus

$$\frac{1}{T_0}\int_0^{T_0} \frac{dG}{dt} dt - \frac{1}{T_0}\int_0^{T_0} \sum_i \alpha_i \frac{d\mathbf{r}_i}{dt} \bullet \mathbf{r}_i dt = 2\overline{T} - n\overline{\mathcal{U}} \, , \tag{1.3.4}$$

where $\overline{\mathcal{U}}$ is the average value of the potential energy for the "non-frictional" forces. Carrying out the integration on the left hand side we have

$$\frac{1}{T_0}\left[G(T_0) - G(0)\right] + \frac{1}{2T_0}\sum_i \alpha_i \left[r_i^2(T_0) - r_i^2(0)\right] = 2\overline{T} - n\overline{\mathcal{U}} \quad . \tag{1.3.5}$$

Thus, if the motion is periodic, both terms on the left hand side of equation (1.3.5) will vanish in a time T_0 equal to the period of the system. Indeed both terms can be made as small as required providing the "frictional" forces $\overline{\mathbf{w}}_i$ do not cause the system to cease to be in motion over the time for which the averaging is done. This apparently academic aside has the significant result that we need not worry about any Lorentz forces or viscosity forces which may be present in our subsequent discussion in which we shall invoke the virial theorem.

4. Continuum-Field Representation of the Virial Theorem

Although nearly all derivations of the virial theorem consider collections of mass-points acting under forces derivable from a potential, it is useful to look at this formalism as it applies to a continuum density field of matter. This is particularly appropriate when one considers applications to stellar structure where a continuum representation of the material is always used.

In the interests of preserving some rigor let us pass from equation (1.2.1) to its analogous representation in the continuum. Let the mass m_i be obtained by multiplying the density $\rho(r)$ by an infinitesimal volume ΔV so that 1.2.1 becomes

$$\mathbf{f}_i = \frac{d}{dt}(\rho \mathbf{v} \Delta V) = \mathbf{v}\Delta V \frac{d\rho}{dt} + \rho \Delta V \frac{d\mathbf{v}}{dt} + \rho \mathbf{v} \frac{d(\Delta V)}{dt} \quad . \tag{1.4.1}$$

Conservation of mass requires that

$$\frac{dm_i}{dt} = \frac{d}{dt}(\rho \Delta V) = \Delta V \frac{d\rho}{dt} + \rho \frac{d(\Delta V)}{dt} = 0 \ . \tag{1.4.2}$$

Multiplying this expression by **v** we see that the first and last terms on the right hand side of equation (1.4.1) are of equal magnitude and opposite sign. Thus, if we define a "force density", ***f***, so that $\boldsymbol{f}\,\Delta V = \mathbf{f}_i$, we can pass to this continuum representation of equation (1.2.1):

$$\boldsymbol{f}(\mathbf{r}) = \rho(\mathbf{r}) \frac{d}{dt}[\mathbf{v}(\mathbf{r})] = \dot{\boldsymbol{p}}(\mathbf{r}) \quad , \tag{1.4.3}$$

where $\boldsymbol{p}(\mathbf{r})$ by analogy to 1.2.1 is just the local momentum density.

We can now define G in terms of the continuum variables so that

$$G = \int_V \boldsymbol{p} \bullet \mathbf{r} dV = \int_V \rho \frac{d\mathbf{r}}{dt} \bullet \mathbf{r} \, dV = \tfrac{1}{2} \int_V \rho \frac{d}{dt}(\mathbf{r} \bullet \mathbf{r}) dV = \tfrac{1}{2} \int_V \rho \frac{dr^2}{dt} dV \quad , \tag{1.4.4}$$

so that

$$G = \tfrac{1}{2} \int_V \frac{d}{dt}(\rho r^2) dV - \tfrac{1}{2} \int_V r^2 \frac{d\rho}{dt} dV \ . \tag{1.4.5}$$

Once again, one uses conservation of mass requiring that the mass within any sub volume V' is constant with time so that $dm(V')/dt = 0$ with that sub-volume V' defined such that

$$\frac{d}{dt}\left(\int_{V'} \rho dV \right) = \int_{V'} \frac{d\rho}{dt} dV = 0 \ . \tag{1.4.6}$$

Thus, the second integral in equation (1.4.5) after integration by parts is zero. If we take the original volume V to be large enough so as to always include all the mass of the object, we may write equation (1.4.5) as

$$G = \frac{1}{2} \frac{d}{dt} \int_V (\rho r^2) dV = \tfrac{1}{2} \frac{dI}{dt} \ . \tag{1.4.7}$$

With these same constraints on V we may differentiate equation (1.4.4) with respect to time and obtain

$$\frac{dG}{dt} = \int_V \left[\boldsymbol{p} \bullet \frac{d\mathbf{r}}{dt} + \mathbf{r} \bullet \frac{d\boldsymbol{p}}{dt} \right] dV = \int_V (\rho v^2 + \mathbf{r} \bullet \boldsymbol{f}) \, dV . \tag{1.4.8}$$

The first term under the integral is just kinetic energy density and hence its volume integral is .just the total kinetic energy of the configuration and

$$\tfrac{1}{2} \frac{d^2 I}{dt^2} = 2T + \int_V \boldsymbol{f} \bullet \mathbf{r} \, dV . \tag{1.4.9}$$

Considerable care must be taken in evaluating the second term in equation (1.4.9) which is basically the virial of Claussius. In the previous derivation we went to some length [i.e., equation (1.2.10)] to avoid "double counting" the forces by noting that the force between any two particles A and B can be viewed as a force at A due to B, or a force at B resulting from A. The contributions to the virial, however, are not equal as they involve a 'dot' product with the position vector. Thus, we explicitly paired the forces and arranged the sum so pairs of particles were only counted once. Similar problems confront us within continuum derivation. Thus, each force at a field point $\boldsymbol{f}(\mathbf{r})$ will have an equal and opposite counterpart at the source points $\mathbf{r}'$.

After some algebra, direct substitution of the potential gradient into the definition of the Virial of Claussius yields [1.2]

$$\int_V \boldsymbol{f} \bullet \mathbf{r}\,dV = -\tfrac{n}{2}\int_V\int_{V'} \rho(\mathbf{r})\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')\bullet(\mathbf{r}-\mathbf{r}')(|\mathbf{r}-\mathbf{r}'|)^{n-2}\,dV'\,dV$$
$$= n\left[\tfrac{1}{2}\int_V\int_{V'} \rho(\mathbf{r})\rho(\mathbf{r}')(|\mathbf{r}-\mathbf{r}'|)^{n}\,dV'\,dV\right] \quad . \qquad 1.4.10$$

Since V = V', the integrals are fully symmetric with respect to interchanging primed with non-primed variables. In addition the double integral represents the potential energy of $\rho(\mathbf{r})$ with respect to $\rho(\mathbf{r}')$, and $\rho(\mathbf{r}')$ with respect to $\rho(\mathbf{r})$; it is just twice the total potential energy. Thus, we find that the virial has the same form as equation (1.2.12), namely,

$$\int_V \boldsymbol{f} \bullet \mathbf{r}\,dV = -n\mathcal{U} \quad . \qquad 1.4.11$$

Substitution of this form into equation (1.4.9) and taking n = -1 yields the same expression for Lagrange's identity as was obtained in equation (1.2.13), specifically,

$$\tfrac{1}{2}\frac{d^2I}{dt^2} = 2T + \Omega \qquad 1.4.12$$

Thus Lagrange's identity, the virial theorem and indeed the remainder of the earlier arguments, are valid for the continuum density distributions as we might have guessed.

Throughout this discussion it was tacitly assumed that the forces involved represented "gravitational" forces insofar as the force was $-\rho\nabla\Phi$. Clearly, if the force depended on some other property of the matter (e.g., the charge density, $\varepsilon(r)$ the evaluation of $\int_V \boldsymbol{f} \bullet \mathbf{r}\,dV$ would go as before with the result that the virial would again be $-n\mathcal{U}$ where $\mathcal{U}$ is the total potential energy of the configuration.

5. The Ergodic Theorem and the Virial Theorem

Thus far, with the exception of a brief discussion in Section 2, we have developed Lagrange's identity in a variety of ways, but have not rigorously taken that finial step to produce the virial theorem. This last step involves averaging over time and it is in this form that the theorem finds its widest application. However, in astrophysics few if any investigators live long enough to perform the time-averages for which the theorem calls. Thus, one more step is needed. It is this step which occasionally leads to difficulty and erroneous results. In order to replace the time averages with something observable, it is necessary to invoke the ergodic theorem.

The Ergodic Theorem is one of those fundamental physical concepts like the Principle of Causality which are so "obvious" as to appear axiomatic. Thus they are rarely discussed in the physics literature. However, to say that the ergodic theorem is obvious is to belittle an entire area of mathematics known as ergodic theory which uses the mathematical language of measure theory. This language alone is enough to hide it forever from the eye of the average physical scientist. Since this theorem is central to obtain what is commonly called the virial theorem, it is appropriate that we spend a little time on its meaning. As noted in the introduction, the distinction between an ensemble average and an average of macroscopic system parameters over time was not clear at the time of the formulation of the virial theorem. However, not too long after, Ludwig Boltzmann[6] formulated an hypothesis which suggested the criterion under which ensemble and phase averages would be the same. Maxwell later stated it this way: *"The only assumption which is necessary for a direct proof is that the system if left to itself in its actual state of motion will, sooner or later, pass through every phase which is consistent with the equation of energy".*[7]

Essentially this constitutes what is most commonly meant by the ergodic theorem. Namely, if a dynamic system passes through every point in phase space then the time average of any macroscopic system parameter, say Q, is given by

$$< Q >_t = \underset{T \to \infty}{\text{Lim}} \left[\frac{1}{T} \right] \int_{t_0}^{t_0+T} Q(t)dt =< Q >_s \quad , \qquad 1.5.1$$

where $<Q>_s$ is some sort of instantaneous statistical average of Q over the entire system.

The importance of this concept for statistical mechanics is clear. Theoretical considerations predict $<Q>_s$ whereas experiment provides something which might be construed to approximately $<Q>_t$. No matter how rapid the measurements of something like the pressure or temperature of the gas, it requires a time which is long compared to characteristic times for the system. The founders of statistical mechanics, such as Boltzmann, Maxwell and Gibbs, realized that such a statement as equation (1.5.1) was necessary to enable the comparison of theory with experiment and thus a great deal of effort was expended to show or at least define the conditions under which dynamical systems were ergodic (i.e., would pass through every point in phase space).

Indeed, as stated, the ergodic theorem is false as was shown independently in 1913 by Rosentha1[8] and Plancherel [9] more modern version of this can be seen easily by noting that no system trajectory in phase space may cross itself. Thus, such a curve may have no multiple points. This is effectively a statement of system boundary conditions uniquely determining the system's past and future. It is the essence of the Louisville theorem of classical mechanics. Such a curve is topologically known as a Jordan curve and it is a well known topological theorem that a Jordan curve cannot pass through all points of a multi-dimensional space. In the language of measure theory, a multi-dimensional space filling curve would have a measure equal to the space whereas a Jordan curve being one-dimensional would have measure zero. Thus, the ergodic hypothesis became modified as the quasi-ergodic hypothesis. This modification essentially states that although a single phase trajectory cannot pass through every point in phase space, it may come arbitrarily close to any given point in a finite time. Already one can sense confusion of terminology beginning to mount. Ogorodnikov[10] uses the term quasi-ergodic to apply to systems covered by the Lewis theorem which we shall mention later. At this point in time the mathematical interest in ergodic theory began to rise rapidly and over the next several years attracted some of the most, famous mathematical minds of the 20th century. Farquhar[11] points out that several noted physicists stated without justification that all physical systems were quasi-ergodic. The stakes were high and were getting higher with the development of statistical mechanics and the emergence of quantum mechanisms as powerful physical disciplines. The identity of phase and time averages became crucial to the comparison of theory with observation.

Mathematicians largely took over the field developing the formidable literature currently known as ergodic theory; and they became more concerned with showing the existence of the averages than with their equality with phase averages. Physicists, impatient with mathematicians for being unable to prove what appears 'reasonable', and also what is necessary, began to require the identity of phase and time averages as being axiomatic. This is a position not without precedent and a certain pragmatic justification of expediency. Some essentially adopted the attitude that since thermodynamics “works”, phase and times averages must be equal. However, as Farquhar observed *“such a pragmatic view reduces statistical mechanics to an ad hoc technique unrelated to the rest of physical theory.”* [12]

Over the last half century, there have been many attempts to prove the quasi-ergodic hypothesis. Perhaps the most notable of which are Birkhoff's theorem[13] and the generalization of a corollary known as Lewis' theorem.[14] These theorems show the existence of time averages and their equivalence to phase averages under quite general conditions. The tendency in recent years has been to bypass phase space filling properties of a dynamical system and go directly to the identification of the equality of phase and time averages. The most recent attempt due to Siniai[15], as recounted by Arnold and Avez[16] proves that the Boltzmann-Gibbs conjecture is correct. That is, a "gas" made up of perfectly elastic spheres confined by a container with perfectly reflecting walls is ergodic in the sense that phase and time averages are equal.

At this point the reader is probably wondering what all this has to do with the virial theorem. Specifically, the virial theorem is obtained by taking the time average of Lagrange's identity. Thus

$$\underset{T\to\infty}{\text{Lim}}\left[\frac{1}{2}\right]\int_{t_0}^{t_0+T}\left(\frac{d^2I}{dt^2}\right)dt = <2T>_t - <\mathcal{U}>_t \quad , \qquad 1.5.2$$

and for systems which are stable the left hand side is zero. The first problem arises with the fact that the time average is over infinite time and thus operationally difficult to carryout [1.3]. Farquhar[17] points out that the time interval must at least be long compared to the relaxation time for the system and in the event that the system crossing time is longer than the relaxation time, the integration in equation(1.5.2) must exceed that time if any statistical validity is to be maintained in the analysis of the system. It is clear that for stars and star-like objects these conditions are met. However, in stellar dynamics and the analysis of stellar systems they generally are not. Indeed, in this case, the astronomer is in the envious position of being in the reverse position from the thermodynamicists. For all intents and purposes he can perform an 'instantaneous' ensemble average which he wishes to equate to a 'theoretically determined' time average. This interpretation will only be correct if the system is ergodic in the sense of satisfying the 'quasi-ergodic hypothesis'. Pragmatically if the system exhibits a large number of degrees of freedom then persuasive arguments can be made that the equating of time and phase averages is justified. However, if isolating integrals of the motion exist for the system, then it is not justified, as these integrals remove large regions of phase space from the allowable space of the system trajectory. Lewis' theorem allows for ergodicity in a sub-space but then the phase averages must be calculated differently and this correspondence to the observed ensemble average is not clear. Thus, the application of the virial theorem to a system with only a few members and hence a few degrees of freedom is invalid unless care is taken to interpret the observed ensemble averages in light of phase averages altered by the isolating integrals of the motion. Furthermore, one should be most circumspect about applying the virial theorem to large systems like the galaxy which appear to exhibit quasi-isolating integrals of the motion. That is, integrals which appear to restrict the system motion in phase space over several relaxation times. However, for stars and star-like objects exhibiting 10^{50} or more particles undergoing rapid collisions and having short relaxation times, these concerns do not apply and we may confidently interchange time and phase averages as they appear in the virial theorem. At least we may do it with the same confidence of the thermodynamicist. For those who feel that the ergodic theorem is still "much ado about nothing", it is worth observing that by attempting to provide a rational development between dynamics and thermodynamics, ergodic theory must address itself to the problems of irreversible processes. Since classical dynamics is fully reversible and thermodynamics includes processes which are not, the nature of irreversibility must be connected in some sense to that of ergodicity and thus to the very nature of time itself. Thus, anyone truly interested in the foundations of physics cannot dismiss ergodic theory as mere mathematical 'nit-picking'.

6. Summary

In this chapter, I have tried to lay the groundwork for the classical virial theorem by first demonstrating its utility, then deriving it in several ways and lastly, examining an important premise of its application. An underlying thread of continuity can be seen in all that follows comes from the Boltzmann transport equation. It is a theme that will return again and again throughout this book. In section 1, we sketched how the Boltzmann transport equation yields a set of conservation laws which in turn supply the basic structure equations for stars. This sketch was far from exhaustive and intended primarily to show the informational complexity of this form of derivation. Being suitably impressed with this complexity, the reader should be in an agreeable frame of mind to consider alternative approaches to solving the vector differential equations of structure in order to glean insight into the behavior of the system. The next two sections were concerned with a highly classical derivation of the virial theorem with section 2 being basically the derivation as it might have been presented a century ago. Section 3 merely updated this presentation so that the formalism may be used within the context of more contemporary field theory. The only 'tricky' part of these derivations involves the 'pairing' of forces. The reader should make every effort to understand or conceptualize how this occurs in order to understand the meaning of the virial itself. The assumption that the forces are derivable from a potential which is described by a power law of the distance alone, dates back at least to Jacobi and is often described as a homogeneous function of the distance.

In the last section, I attempted to provide some insight into the meaning of a very important theorem generally known as the ergodic theorem. Its importance for the application of the virial theorem cannot be too strongly emphasized. Although almost all systems of interest in stellar astrophysics can truly be regarded as ergodic, many systems in stellar dynamics cannot. If they are not, one cannot replace averages over time by averages over phase or the ensemble of particles without further justification.

Notes to Chapter 1

1.1 Since $a_{ij} = a_{ji}$ for all known physical forces, we may substitute equation (1.2.9) in equation (1.2.11) as follows:

$$\sum_i \mathbf{f}_i \bullet \mathbf{r}_i = -\sum_i \sum_{j>i} n a_{ij} r_{ij}^{(n-2)} \left[(\mathbf{r}_i - \mathbf{r}_j) \bullet \mathbf{r}_i + (\mathbf{r}_j - \mathbf{r}_i) \bullet \mathbf{r}_j \right]$$
$$= -n \sum_i \sum_{j>i} a_{ij} r_{ij}^{(n-2)} \left[(\mathbf{r}_i - \mathbf{r}_j) \bullet (\mathbf{r}_i - \mathbf{r}_j) \right] = -n \sum_i \sum_{j>i} a_{ij} r_{ij}^{(n-2)} r_{ij}^2 \quad . \qquad \text{N 1.1.1}$$

Thus

$$\sum_i \mathbf{f}_i \bullet \mathbf{r}_i = -n \sum_i \sum_{j>i} a_{ij} r_{ij}^n = -n \sum_i \sum_{j>i} \Phi(r_{ij}) \quad . \qquad \text{N 1.1.2}$$

Since the second summation is only over j > i, there is no "double-counting" involved, and the double sum is just the total potential energy of the system.

1.2 As in Section 2, let us assume that the force density is derivable from a potential which is a homogeneous function of the distance between the source and field point.[5] Then, we can write the potential as

$$\Phi(\mathbf{r}) = \int_{V'} \rho(\mathbf{r}) \left(|\mathbf{r} - \mathbf{r}'| \right)^n dV' \quad \forall\ n < 0\,, \qquad \text{N 1.2.1}$$

and the force density is then

$$\boldsymbol{f}(\mathbf{r}) = -\rho(\mathbf{r}) \nabla_r \Phi(\mathbf{r}) = -\rho(\mathbf{r}) \int_{V'} \rho(\mathbf{r}') \nabla_r \left(|\mathbf{r} - \mathbf{r}'| \right)^n dV' \quad , \qquad \text{N 1.2.2}$$

while the force density at a source point due to all the field points is

$$\boldsymbol{f}(\mathbf{r}') = -\rho(\mathbf{r}') \nabla_{r'} \Phi(\mathbf{r}') = -\rho(\mathbf{r}') \int_{V} \rho(\mathbf{r}) \nabla_{r'} \left(|\mathbf{r} - \mathbf{r}'| \right)^n dV\,, \qquad \text{N 1.2.3}$$

where ∇_r and $\nabla_{r'}$ denote the gradient operator evaluated at the field point **r** and the source point **r'** respectively. Since the contribution to the force density from any pair of sources and field points will lie along the line joining the two points,

$$\nabla_r \left(|\mathbf{r} - \mathbf{r}'| \right)^n = -\nabla_{r'} \left(|\mathbf{r} - \mathbf{r}'| \right)^n = n \left(|\mathbf{r} - \mathbf{r}'| \right)^{n-2} (\mathbf{r} - \mathbf{r}') \quad . \qquad \text{N 1.2.4}$$

Now $\int_V \boldsymbol{f}(\mathbf{r}) \bullet \mathbf{r} dV = \int_{V'} \boldsymbol{f}(\mathbf{r}') \bullet \mathbf{r}' dV'$, so multiplying equation (N 1.2.2) by **r** and integrating over V produces the same result as multiplying equation (N 1.2.3) by **r'** and integrating over V'. Thus, doing this and adding equation (N 1.2.2) to equation (N 1.2.3). we get

$$2 \int_V \boldsymbol{f} \bullet \mathbf{r} dV = -\int_V \rho(\mathbf{r}) \int_{V'} \rho(\mathbf{r}) \mathbf{r} \bullet \nabla_r \left(|\mathbf{r} - \mathbf{r}'| \right)^n dV' dV = -\int_{V'} \rho(\mathbf{r}') \int_V \rho(\mathbf{r}) \mathbf{r} \nabla_{r'} \left(|\mathbf{r} - \mathbf{r}'| \right)^n dV' dV \,. \qquad \text{N 1.2.5}$$

1.3 It should be noted that the left hand side of 1.5.2 is zero if the system is periodic and the integral is taken over the period.

References

1 Kurth, Rudolf (1957), *Introduction to the Mechanics of Stellar Systems*, Pergamon Press, N. Y., London, Paris, pp. 69.

2 Landau, L. D. & Lifshitz, E. M. (1962), *The Classical Theory of Fields*, Trans. M. Hamermesh, Addison-Wesley Pub. Co., Reading, Mass. USA, p. 95-97.

3 Chandrasekhar, S. (1957), *An Introduction to the Study of Stellar Structure*, Dover Pub., Inc., pp. 49-51.

4 Goldstein, H. (1959), *Classical Mechanics*, Addison-Wesley Pub. Co., Reading, Mass. , pp. 69-71.

5 Landau, L. D. & Lifshitz, E. M. (1960), *Mechanics*, Trans. J. B. Sykes & J. S. Bell, Addison-Wesley Publ. Co., Reading, Mass., pp. 22-23.

6 Boltzmann, L. (1811), Sitzler Akad. Wiss. Wien 63, 397, 679.

7 Maxwell, J. C. (1879), Trans. Cam. Phil. Soc. 12, p.547.

8 Rosenthal, A. (1913), Ann. der Physik 42, p.796.

9 Plancherel, M. (1913), Ann. der Physik 42, p.1061.

10 Ogorodnikov, K. F. (1965), *Dynamics of Stellar Systems*, Pergamon Press, The McMillan Co., New York, p. 153.

11 Farquhar, I. E. (1964), *Ergodic Theory in Statistical Mechanics*, Interscience Pub., John Wiley & Sons, Ltd., London, New York, Sydney, p. 77.

12 .ibid., p. 3.

13 Birkhoff, G. D. (1931), Proc. of Nat'l. Acad. of Sci.: U.S. 17, p. 656.

14 Lewis, R. M. (1966), Arch. Rational Mech. Anal 5, p. 355.

15 Sinai, Ya. (1962), Vestnik Mossovskova Gosudrastvennova Universitata Series Math. 5.

16 Arnold, V. I., and Avez, A. (1968), *Ergodic Problem of Classical Mechanics*, W. A. Benjamin, Inc., New York, Amsterdam, p. 78.

17 Farquhar, I. E. (1964), Loc. cit. pp. 23-32.