II Contemporary Aspects of the Virial Theorem

1. The Tensor Virial Theorem

The tensor representation of the virial theorem is an attempt to restore some of the information lost in reducing the full vector equations of motion described in Chapter I, section 1 to scalars. Although the germ of this idea can be found developing as early as 1903 in the work of Lord Rayleigh[1], it wasn't until the 1950's that Parker[2, 3], and Chandrasekhar and Fermi[4] found the concept particularly helpful in dealing with the presence of magnetic fields. The concept was further expanded by Lebovitz[5] and in a series of papers by Chandrasekhar and Lebovitz[6, 7] during the 1960's, for the investigation of the stability of various gaseous configurations. *Chandrasekhar*[33] *has given a fairly comprehensive recounting of his efforts on this subject after the original version of this monograph was prepared*. However, the most lucid derivation is probably that presented by Chandrasekhar[8] in 1961 and it is a simplified version of that derivation which I shall give here.

As previously mentioned the motivation for this approach is to regain some of the information lost in forming the scalar virial theorem by keeping track of certain aspects of the system associated with its spatial symmetries. If one recalls the full-blown vector equations of motion in Chapter I, section 1, this amounts to keeping some of the component information of those equations, but not all. In particular, it is not surprising that since system symmetries inspire this approach that the information to be kept relates to motions along orthogonal coordinate axes.

At this point, it is worth pointing out that the derivation in Chapter I, section 2, essentially originates from the equations of motion of the system being considered. The derivations take the form of multiplying those equations of motions by position vectors and averaging over the spatial volume. The final step involves a further average over time. That is to say that the virial theorem results from taking spatial moments of the equations of motion and investigating their temporal behavior. (Recall that the equations of motion themselves are moments of the Boltzmann transport equation.) Since moment analysis of this type also yields some of the most fundamental conservation laws of physics (i.e., momentum, mass and energy), it is not surprising that the virial theorem should have the same power and generality as these laws. Indeed, it is rather satisfying to one who believes that "all that is good and beautiful in physics" can be obtained from the

Boltzmann equation that the virial theorem essentially arises from taking higher order moments of that equation. With that in mind let us consider a collisionless pressure-free system analogous with that considered in Chapter I, section 2 and neglect viscous forces and macroscopic forces such as net rotation and magnetic fields as we shall consider them later. Under these conditions, equation (1.1.4) becomes

$$\rho\frac{\partial\mathbf{u}}{\partial t}+\rho(\mathbf{u}\bullet\nabla)\mathbf{u}=-\rho\nabla\Phi=\rho\frac{d\mathbf{u}}{dt} \quad , \qquad 2.1.1$$

which is simply the vector representation of either equations (1.1.4) or (1.2.1). In Chapter I, section 2, we essentially took the inner product of equation (1.1.4) with the position vector **r** and integrated over the volume to produce a scalar equation. Here we propose to take the outer product of equation (1.1.1) with the position vector **r** producing a tensor equation which can be regarded as a set of equations relating the various components of the resulting tensors. Cursory dimensional arguments should persuade one that this procedure should produce relationships between the various moments of inertia of the system and energy-like tensors. Thus, our starting point is

$$\int_V \rho\mathbf{r}\frac{d\mathbf{u}}{dt}dV=\int_V \rho\mathbf{r}\nabla\Phi dV \quad . \qquad 2.1.2$$

Although some authors choose a slightly different convention, the term on the right hand side of equation (2.1.2) can properly be called the virial tensor. Now, as before let the potential be

$$\Phi(\mathbf{r})=\int_{V'}\rho(\mathbf{r}')(|\mathbf{r}-\mathbf{r}'|)^n dV' \qquad \forall n<0 \quad . \qquad 2.1.3$$

Then following exactly the same manipulation as in Chapter I, only taking into account outer products instead of inner products with the position vectors, we get the virial tensor.[2.1]

$$\int_V \rho\mathbf{r}\nabla\Phi dV = n\left\{\tfrac{1}{2}\int_V\int_{V'}\rho(\mathbf{r})\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')(\mathbf{r}-\mathbf{r}')(|\mathbf{r}-\mathbf{r}'|)^{n-2}dV'dV\right\}. \qquad 2.1.4$$

If we define

$$\begin{aligned}\mathfrak{I}&=\int_V(\rho\mathbf{r}\mathbf{r})dV\\ \mathfrak{T}&=\tfrac{1}{2}\int_V(\rho\mathbf{u}\mathbf{u})dV \quad , \qquad 2.1.5\\ \mathfrak{U}&=\tfrac{1}{2}\int_V\int_{V'}\rho(\mathbf{r})\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')(\mathbf{r}-\mathbf{r}')(|\mathbf{r}-\mathbf{r}'|)^{n-2}dV'dV\end{aligned}$$

equation (2.1.2 becomes)

$$\tfrac{1}{2}\frac{d^2\mathfrak{I}}{dt^2}=2\mathfrak{T}+n\mathfrak{U}. \qquad 2.1.6$$

which is essentially the tensor representation of Lagrange's identity[2.2] where $\mathfrak{I}$ is sometimes called the moment of inertia tensor, $\mathfrak{T}$ the kinetic energy tensor and $\mathfrak{U}$ the potential energy tensor. By eliminating additional external forces such as magnetic fields and rotation we have lost much of the power of the tensor approach. However, some insight into this power can be seen by considering in component form one term in the expansion of the virial tensor[2.1].

$$\int_V \rho \frac{d}{dt}\left(\mathbf{r}\frac{d\mathbf{r}}{dt}\right)dV = \int_V \rho \frac{d}{dt}\left(x_i \frac{dx_j}{dt}\right)dV \quad . \qquad 2.1.7$$

Since this tensor is clearly symmetric we find, by using the same conservation of mass arguments discussed earlier, that

$$\frac{d}{dt}\int_V \rho\left(x_i \frac{dx_j}{dt} - x_j \frac{dx_i}{dt}\right)dV = 0 \ , \qquad 2.1.8$$

which simply says the angular momentum about x_k is conserved. Thus the tensor virial theorem leads us to a fundamental conservation law which would not have been apparent from the scalar form derived earlier.

2 Higher Order Virial Equations

In the last section it became clear that both the scalar and tensor forms of the virial theorem are obtained by taking spatial moments of the equations of motion. Chandrasekhar[9] was apparently the first to note this and to inquire into the utility of taking higher moments of the equations of motion. There certainly is considerable precedent for this in mathematical physics. As already noted, moments in momentum-space of the Boltzmann transport equation yield expressions for the conservation of mass, momentum and energy. Spatial moments of the transport equation of a photon gas can be used to obtain the equation of radiative transfer. Approximate solutions to the resulting equations can be found if suitable assumptions such as the existence of an equation of state are made to "close" the moment equations. Such is the origin of such diverse expressions as the Eddington approximation in radiative transfer, the diffusion approximation in radiative transfer, the diffusion approximation in gas dynamics and many others. Usually, the higher the order of the moment expressions, the less transparent their physical content. Nevertheless, in the spirit of generality, Chandrasekhar investigated the properties of the first several moment equations. In a series of papers, Chandrasekhar and Lebovitz[10, 11] and later Chandrasekhar[12, 13] developed these expressions as far as the fourth-order moments of the equations of motion.

Since for no moment expressions other than those of the first moment do any terms ever appear that can be identified with the Virial of Claussius, it is arguable as to whether they should be called virial expressions at all. However, since it is clear that this investigation was inspired by studies of the classical virial theorem, I will briefly review their development. Recall the Euler-Lagrange equation of hydrodynamic flow developed in Chapter I, equation (1.1.4)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \bullet \nabla)\mathbf{u} = -\nabla\Psi - \frac{1}{\rho}\nabla \bullet \mathfrak{P} - \frac{1}{\rho}\int \mathbf{S}(\mathbf{v}-\mathbf{u})dv \, . \qquad 2.2.1$$

Quite simply the nth order "virial equations" of Chandrasekhar are generated by taking (n-l) outer tensor products of the radius vector **r** and equation (2.2.1). The result is then integrated over all physical space. This leads to a set of tensor equations containing tensors of rank n. If we assume that particle collisions are isotropic, then the source term of equation (2.2.1) vanishes and $\nabla \bullet \mathfrak{P} = \nabla P$. The symbolic representation for the nth order "virial equation" can then be written as:

$$\int_V \mathbf{r}^{(n-1)} \rho \frac{d\mathbf{u}}{dt} dV + \int_V \mathbf{r}^{(n-1)} \rho \nabla \Phi + \int_V \mathbf{r}^{(n-1)} \nabla P dV = 0 \,. \tag{2.2.2}$$

Recalling our arguments in Chapter I, section 4, about conservation of mass, it follows from equation (1.4.6) that

$$\frac{d}{dt}\left(\int_V \rho Q dV \right) = \int_V \rho \frac{dQ}{dt} dV \quad , \tag{2.2.3}$$

where Q is any point-defined property of the medium. Thus, the first order "virial equations", correspond to the integrals of the equation of motion themselves over volume. So

$$\frac{d}{dt}\int_V \rho \mathbf{u} dV + \int_V \rho \nabla \Phi dV + \int_V \nabla P dV = 0 \quad . \tag{2.2.4}$$

Noting that $\nabla Q = \nabla \bullet (\mathbb{1} Q/3)$, the second and third integrals are zero by the divergence theorem and equation (2.2.4) becomes

$$\tfrac{1}{2}\frac{d^2}{dt^2}\left(\int_V \rho \mathbf{r} dV \right) = 0 \,, \tag{2.2.5}$$

essentially telling us that the center of mass $\left(\int_V \rho \mathbf{r} dV \right)$ is not being accelerated. Setting n = 2 we arrive at the second order "virial equations" that are the tensor version of Lagrange's identity that we discussed in the previous section.

As in equation (2.2.2), we can represent the nth order "virial equations" as

$$\int_V \rho \mathbf{r}^{(n-1)} \frac{d\mathbf{u}}{dt} dV + \int_V \rho \mathbf{r}^{(n-1)} \nabla \Phi dV + \int_V \mathbf{r}^{(n-1)} \nabla P dV = 0 \quad , \tag{2.2.6}$$

which after use of continuity and the divergence theorem becomes

$$\frac{d}{dt}\int_V \rho \mathbf{r}^{(n-1)} \mathbf{u} dV - \int_V \rho \frac{d}{dt}\left(\mathbf{r}^{(n-1)}\right)\mathbf{u} dV + \int_V \rho \mathbf{r}^{(n-1)} \nabla \Phi dV + \int_V P \mathbb{1} \bullet \nabla \left(\mathbf{r}^{(n-1)}\right) dV = 0 \,. \tag{2.2.7}$$

Since the outer product in general does not commute, the integrals of the second and fourth term become strings of tensors of the form

$$\rho\frac{d}{dt}\left(\mathbf{r}^{(n-1)}\right)\mathbf{u} = \mathbf{u}\left(\mathbf{r}^{(n-2)}\right)\mathbf{u} + \mathbf{r}\mathbf{u}\left(\mathbf{r}^{(n-3)}\right)\mathbf{u} + \cdots + \mathbf{r}^{(n-2)}\mathbf{u}\mathbf{u}\ ,$$

and 2.2.8

$$P\nabla\left(\mathbf{r}^{(n-1)}\right) = P\left(\mathbb{1}\mathbf{r}^{(n-2)} + \mathbf{r}\mathbb{1}\mathbf{r}^{(n-3)} + \cdots + \mathbf{r}^{(n-2)}\mathbb{1}\right) \quad .$$

However, the first term can be written as

$$\frac{d}{dt}\int_V \rho\mathbf{r}^{(n-1)}\mathbf{u}dV = \frac{1}{n!}\frac{d^2}{dt^2}\int_V \rho\mathbf{r}^{(n)}dV \quad . \qquad 2.2.9$$

Thus, each term in equation (2.2.7) represents one or more tensors of rank n, the first of which is the second time derivative of a generalized moment of inertia tensor and the last three are all 'energy-like' tensors. From equation (2.2.8), it is clear that the second integral will generate tensors which are spatial moments of the kinetic energy distributions while the last term will produce moment tensors of the pressure distribution. The third integral is, however, the most difficult to rigorously represent. For n = 2 we know it is just the total potential energy. Chandrasekhar[8] shows how these tensors can be built up from the generalized Newtonian tensor potential or alternatively from a series of scalar potentials which obey the equations

$$\begin{aligned} \nabla^2\Psi &= -4\pi G\rho \\ \nabla^4\Im &= -8\pi G\rho \\ \nabla^6\Re &= -32\pi G\rho \\ &\vdots \end{aligned} \quad . \qquad 2.2.10$$

Thus, we have formulated a representation of what Chandrasekhar calls the "higher order virial equations". They are, in fact, spatial tensor moments of the equations of motion. We may expect them to be of importance in the same general way as the virial theorem itself. That is, in stationary systems the left hand side of equation (2.2.7) vanishes and the result is a system of identities between the various tensor energy moments. Keeping in mind that any continuum function can be represented in terms of a moment expansion, equation (2.2.7) must thus contain all of the information concerning the structure of the system. These equations thus represent an alternate form to the solution of the equation of motions. Like most series expansions, it is devoutly to be wished that they will converge rapidly and the "higher order" tensors can usually be neglected.

3. Special Relativity and the Virial Theorem

So far we have considered only the virial theorem that one obtains from the Newtonian equations of motion. Since there are systems such as white dwarfs, wherein the dynamic pressure balancing gravity is supplied by particles whose energies are very much larger than their rest energy, it is appropriate that we investigate the extent to which we shall have to modify the virial theorem to include the effects of special relativity. For systems in equilibrium, the virial theorem says $2T = \Omega$. One might say that it requires a potential energy equal to 2T to confine the motions of particles having a total kinetic energy T. As particles approach the velocity of light the kinetic energy increases without bound. One may interpret this as resulting from an unbounded increase of the particle's mass. This increase will also affect the gravitational potential energy, but the effect is quadratic. Thus we might expect in a relativistic system that a potential energy less than 2T would be required to maintain equilibrium. This appears to be the conclusion arrived at by Chandrasekhar when, while investigating the internal energy of white dwarfs he concludes that as the system becomes more relativistically degenerate, T approaches Ω and this "*must be the statement of the virial theorem for material particles moving with very nearly the velocity of light*."[14]

This is indeed the case and is the asymptotic limit represented by a photon gas or polytrope of index $n = 3$ (see Collins[32]). In order to obtain the somewhat more general result of a relativistic form of Lagrange's identity, we shall turn to the discussion of relativistic mechanics of Landau and Lifshitz[15] mentioned in Chapter I. As most discussions in field mechanics generally start from a somewhat different prospective, let us examine the correspondence with the starting point of the equations of motion adopted in the earlier sections. Generally most expositions of field mechanics start with the statement that

$$\Box \bullet \Im = 0\,, \qquad 2.3.1$$

where $\Im$ is the Maxwell stress-energy tensor and $\Box$ is known as the *four-gradient operator*. This is equivalent to saying that there exists a volume in space-time sufficiently large so that outside that volume the stress-energy tensor is zero. This equivalence is made obvious by applying Gauss' divergence theorem so that

$$\int_R \Box \bullet \Im dR = \int_S \Im \bullet d\mathbf{s} = 0\,. \qquad 2.3.2$$

In short, equation (2.3.1) is a conservation law. We have already seen that the fundamental conservation laws of physics are derivable from the Boltzmann transport equation as are the equations of motion. Indeed, the operation of taking moments is quite similar in both cases. Thus, both starting points are equivalent as they have their origin in a common concept.

Although the conceptual development for this derivation is inspired by Landau and Lifshitz the subscript notation will be largely that employed by Misner, Thorne, and Wheeler[16]. Tempting as it is to use the coordinate free geometry of these authors, the concept of taking moments at this point is most easily understood within the context of a coordinate representation so for the moment we will keep that approach. In a Lorentz coordinate system, Landau and Lifshitz give components of the 4-velocity of a particle as

$$u_\alpha = \frac{dx_\alpha}{ds}. \quad \alpha = 0 \cdots 3 \quad , \qquad 2.3.3$$

where $ds/dt = c(1 - v^2/c^2)^{1/2} = \gamma c$. *Note this is a somewhat unconventional definition of* γ.

The components of the energy-momentum tensor are

$$\Im_{\alpha\beta} = \rho c u_\alpha u_\beta \left(ds/dt \right) \quad , \qquad 2.3.4$$

and specifically $\Im_{0j} = i\rho c^2 u_j$,

which are clearly symmetric in α and β. Since $\sum_{\alpha=0}^{3} u_\alpha^2 = -1$, the trace of equation (2.3.4) is

$$\sum_{\alpha=0}^{3} \Im_{\alpha\alpha} = -\rho c^2 \gamma \quad . \qquad 2.3.5$$

In terms of the three-dimensional components the conservation law expressed by equation (2.3.1) can be written as

$$\frac{1}{ic}\frac{\partial \Im_{j0}}{\partial t} + \sum_{k=1}^{3} \frac{\partial \Im_{jk}}{\partial x_k} = c\frac{\partial(\rho u_j)}{\partial t} + \sum_{k=1}^{3} \frac{\partial \Im_{jk}}{\partial x_k} = 0 \, . \qquad 2.3.6$$

Substituting for the components of $\Im$ and repeating the algebra of earlier derivations we have[2.3]

$$\sum_{j=1}^{3} \frac{d}{dt} \int_V \frac{\rho}{2} \frac{d}{dt} \left(x_j x_j / \gamma \right) dV = \Omega + T + \int_V \gamma \tau dV \quad , \qquad 2.3.7$$

where τ is the kinetic energy density of relativistic particles[17]. Again, using the conservation of mass arguments in Chapter I, this becomes

$$\sum_{j=1}^{3} \tfrac{1}{2} \frac{d^2}{dt^2} \int_V \left(\rho x_j x_j / \gamma \right) dV = \Omega + T + \int_V \gamma \tau dV \quad . \qquad 2.3.8$$

where, if we define the volume integral on the left to be the relativistic moment of inertia, we can write

$$\tfrac{1}{2} \frac{d^2}{dt^2} \left(I_r \right) = \Omega + T + \int_V \gamma \tau dV \quad . \qquad 2.3.9$$

In the low velocity limit $\gamma \to 1$ so that $I_r \to I$ and we recover the ordinary Lagrange's identity. In the relativistic limit as $\gamma \to 0$ we recover for stable systems the Chandrasekhar result that $T + \Omega = 0$ (i.e., the total energy $E = 0$). Thus, it is fair to say that equation (2.3.9) is an expression of the Lagrange's identity including the effects of special relativity.

It is worth noting by analogy with section 1 that the tensor relativistic theorem can be derived by taking the outer or 'tensor' product of the space-like position vector with equation (2.3.6) and integrating over the volume. Following the same steps that lead to equation (N.2.3.3), we get

$$c \int_V x_i \frac{\partial(\rho u_j)}{\partial t} dV - \int_V \Im_{ij} dV = 0 \, . \qquad 2.3.10$$

Since $\Im_{ij}$ is symmetric, we can add equation (2.3.10) to its counterpart with the indices interchanged, and get

$$c\int_V \left(x_i \frac{\partial(\rho u_j)}{\partial t} + x_j \frac{\partial(\rho u_i)}{\partial t} \right) dV - 2\int_V \Im_{ij} dV = 0 \,, \qquad 2.3.11$$

or finally

$$\tfrac{1}{2}\frac{d^2}{dt^2}\left(\int_V [\rho(x_j x_i)/\gamma \right) dV = \int_V \Im_{ij} dV \quad . \qquad 2.3.12$$

The integral on the left can be viewed on the relativistic moment of inertia tensor while the right hand side is just the volume integral of the components of the energy momentum tensor. Following the prescription used to generate equation (2.3.11), but subtracting the transpose of equation (2.3.11) from itself, yields

$$\int_V \left(x_i \frac{\partial(\rho u_j)}{\partial t} - x_j \frac{\partial(\rho u_i)}{\partial t} \right) dV = 0 \quad , \qquad 2.3.13$$

which becomes by application of Leibniz's law

$$\frac{d}{dt}\left[\int_V \frac{\rho}{\gamma} \left(x_i \frac{\partial(\rho u_j)}{\partial t} + x_j \frac{\partial(\rho u_i)}{\partial t} \right) dV \right] = 0 \,, \qquad 2.3.14$$

and is the relativistic form of the expression for the conservation of angular momentum obtained in section 1, equation (2.1.8).

4. General Relativity and the Virial Theorem

The development of quantum theory and the formulation of the general theory of relativity probably represent the two most significant advances in physical science in the twentieth century. In light of the general nature and wide applicability of the virial theorem it is surprising that little attempt was made during that time to formulate it within the context of general relativity. Perhaps this was a result of the lack of physical phenomena requiring general relativity for their description or possibly the direction of mathematical development undertaken for theory itself. For the last twenty years, there has been a concerted effort on the part of relativists to seek coordinate-free descriptions of general relativity in order to emphasize the connection between the fundamental geometrical properties of space and the description of associated physical phenomena. Although this has undoubtedly been profitable for the development of general relativity, it has drawn attention away from that technique in theoretical physics known as 'moment analysis'. This technique produces results which are in principle coordinate independent but usually utilize some specific coordinate frames for the purpose of calculation.

Another point of difficulty consists of the nature of the theory itself. General relativity, like so many successful theories, is a field theory and is thus concerned with functions defined at a point. Virtually every version of the virial theorem emphasizes its global nature.† That is, some sort of symmetrical volume is integrated or summed over to produce the appropriate physical parameters. This difference becomes a serious problem when one attempts to assign a physical operational interpretation to the quantities represented by the spatial integrals. The problem of operational definition of macroscopic properties in general relativity has plagued the theory since its formulation. Although continuous progress has been made, there does not exist any completely general formulation of the virial theorem within the framework of general relativity at this time. This certainly is not to say that such a formulation cannot be made. Indeed, what we have seen so far should convince even the greatest skeptic that such a formulation does exist since its origin is basically that of a conservation law (see footnote at the end of the chapter). Even the general theory of relativity recognizes conservation laws although their form is often altered. Let us take a closer look at the origin of some of these problems. This can be done by taking into account in a self-consistent manner in the Einstein field equations all terms of order $1/c^2$. This is the approach adopted by Einstein, Infield, and Hoffman[19] in their approach to the relativistic n-body problem and successfully applied by Chandrasekhar[20] to hydrodynamics. Although more efficient approximation techniques exist for the calculation of higher order relativistic phenomena such as gravitational radiation, this time honored approach is adequate for calculating the first order (i.e. c^2) terms commonly known as the post-Newtonian terms.†† During the first half of 1960s, in studying the hydrodynamics of various fluid bodies, Chandrasekhar developed the virial theorem to an extremely sophisticated level. The most comprehensive recognition of this work can be found in his excellent book on the subject[9]. One of the highlights not dealt with in the book are his efforts to include the first-order effects of general relativity. In an impressive and lengthy paper during 1965, Chandrasekhar developed the post-Newtonian equations of hydrodynamics including a formulation of the virial theorem[20]. It is largely this effort which we shall summarize here. One of the fundamental difficulties with the general theory of relativity is its non-linearity. The physical properties of matter are represented by the geometry of space and in that turn determines the geometry of space. It is this non-linearity that causes so much difficulty with approximation theory and with which the Einstein, Infield, Hoffman theory (EIH) deals directly.

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It is worth noting that in order to 'test' any field theory against observation, it is necessary to compare integrals of the field quantities with the observed quantities. Even something as elementary as density is always "observed" by comparing some mass to some volume. It is impossible in principle to measure anything at a point. This obvious statement causes no trouble as long as we are dealing with concepts well within the range of our experience where we can expect our intuition to behave properly. However, beyond this comfortable realm, we are liable to attribute physical significance and testability to quantities which are in principle untestable. The result is to restrict the range of possibility for a theory unnecessarily.

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For a beautifully concise and complete summary of the post-Newtonian approximation, see Misner, Thorne, and Wheeler, Gravitation (1973), w. H. Freeman & Co., San Francisco, Chapter 39.

The basic approach assumes that the metric tensor can be written as being perturbed slightly from the flat-space or Euclidean metric so that

$$\mathfrak{g}_{\alpha\beta} = \mathfrak{g}^{(0)}_{\alpha\beta} + \mathfrak{h}_{\alpha\beta}\,, \qquad 2.4.1$$

where the $\mathfrak{h}_{\alpha\beta}$ are small terms of the order of c^{-2} or smaller. This enables one to determine the elements of the energy-momentum tensor up to terms of the order of c^{-2} from its definition so that

$$\mathfrak{I}_{\alpha\beta} = (\varepsilon + P)u_\alpha u_\beta - P\mathfrak{g}_{\alpha\beta}\ . \qquad 2.4.2$$

The Einstein field equations can be written in terms of the Ricci tensor and the energy-momentum tensor as

$$\mathfrak{R}_{\alpha\beta} = -\frac{8\pi G}{c^4}\left(\mathfrak{I}_{\alpha\beta} - \tfrac{1}{2}J\mathfrak{g}_{\alpha\beta}\right), \qquad 2.4.3$$

where J is just the trace $\mathfrak{I}_{\alpha\beta}$. The Ricci tensor is essentially a geometric tensor and contains information relating to the metric alone.

The EIH approximation provides a prescription for solving the field equations in various powers of $1/c^2$ given the information concerning $\mathfrak{I}_{\alpha\beta}$ and $\mathfrak{g}_{\alpha\beta}$. In general the procedure determines the metric coefficients to one higher order than was originally specified. This procedure can be repeated but there remain some unsolved problems as to convergence of the scheme in general. At any point one may use the perturbed metric and the prescription for obtaining the equations of motion to generate a set of perturbed equations of motion. The relativistic prescription that free particles follow geodesic paths is logically equivalent to stating the four-space divergence of the energy-momentum tensor is zero. That is

$$\Box \bullet \mathfrak{I}_{\alpha\beta} = 0\,. \qquad 2.4.4$$

Indeed it is this condition that in the flat-space metric yields of the Euler-Lagrange equations of hydrodynamic flow. It was this condition that we needed in section 3 to obtain a form of the virial theorem appropriate for special relativity. Unfortunately the process of taking the divergence looses one order of approximation and thus it is not possible to go directly from the perturbed metric to the equations of motion and maintain the same level of accuracy. One must first pass through the field equations and the EIH approximation scheme. In order to follow this prescription one must first start with an approximation to the metric $\mathfrak{g}_{\alpha\beta}$. Here it is traditional to invoke the principle of equivalence that requires that[21]

$$\mathfrak{h}_{\alpha\beta} = -\frac{2\Psi}{c^2}\delta_{\alpha\beta} + \mathbf{O}(<1/c^3)\,. \qquad 2.4.5$$

With this as a starting point one may proceed with the approximation scheme and obtain the equations of motion. Like many approximation schemes the mathematical manipulations are formidable and physical insight easily lost. However, progress is rapid in a field such as this and what was an original research effort by Chandrasekhar in 1965 becomes a 'homework' problem for Misner, Thorne and Wheeler[16] in 1972 (e.g. M.T.W. exercise 39.13.) Although it is contrary to the spirit of this book to quote results without derivation, I find in this case I must. We have laid neither a sufficient mathematical framework nor developed the general theory of relativity sufficiently to present the derivation in detail without consuming excessive space. Instead, consider the types of effects we might expect general relativity to introduce and see if these can be identified in the resulting equations of motion.

First, energy is matter and therefore its motion must be followed in the equations of motion as well as that of matter. This is really a consequence of special relativity but insofar as this 'added' mass affects the metric, we should find its effects present. The distortion of space also changes or at least complicates what is meant by a volume and thus it is useful to define a density ρ* which obeys a continuity equation

$$\frac{\partial\rho^*}{\partial t} + \nabla\bullet(\rho^*\mathbf{u}) = 0 \quad ,$$

where

$$\rho^* = \rho_0\left[1 + (u^2 + 3\Psi)/c^2\right] \quad . \qquad 2.4.6$$

For purposes of simplification, Chandrasekhar finds it convenient to define a slightly altered form of the density which explicitly contains the internal energy of the gas and has a slightly altered continuity equation

$$\frac{\partial\sigma}{\partial t} + \nabla\bullet\left[\sigma\mathbf{u} + \frac{1}{c^2}\left(\rho_0\frac{\partial\Psi}{\partial t} - \frac{\partial\rho_0}{\partial t}\right)\right] = 0 \, , \qquad 2.4.7$$

where

$$\sigma = \rho_0[1 + (u^2 + 2\Psi + \Pi + P/\rho_0)/c^2] \, .$$

Here Π/c^2 is the internal energy of the gas and P is the local pressure. It is worth noting that ρ_o is the density one would find in the absence of general relativity but where relativistic effects are important it is essentially a non-observable quantity since one can not devise a test for measuring it.

The general theory of relativity is a non-linear theory and thus we should expect terms to appear which reflect this non-linearity. They will be of different types. Firstly one should expect effects of the Newtonian potential Ψ affecting the metric directly which in turn modifies Ψ. These terms are indeed present but Misner, Thorne and Wheeler show that they can be represented by direct integrals over the mass distribution[22]. Secondly, since the matter and the metric are inexorably tied together, motion of matter will 'drag' the metric which will introduce velocity dependent terms in the 'potentials' used to represent those terms.

Both these effects can indeed be represented by 'potentials' but not just the Newtonian potential. Thus, various authors introduce various kinds of potentials to account for these non-linear terms. With this in mind the equations of motion as derived by Chandrasekhar become[23]

$$\frac{\partial(\sigma\mathbf{u})}{\partial t} - \rho_0\nabla\Psi + \nabla\left[\left(1 + 2\Psi/c^2\right)P\right] + \frac{\rho_0}{c^2}\left\{\frac{d}{dt}\left[4\mathbf{u}\Psi - \tfrac{7}{2}\mathbf{Y} - \tfrac{1}{2}\nabla\left(|\mathbf{Y}|\right)\right] + 4\mathbf{u}\bullet(\nabla\mathbf{Y}) - \tfrac{1}{2}(\mathbf{u}\bullet\nabla)[\mathbf{Y} - \tfrac{1}{2}\nabla(|\mathbf{Y}|)]\right\}$$
$$+ \frac{1}{2\pi Gc^2}\left[\nabla^2\Phi\nabla\Psi + \nabla^2\Psi\nabla\Phi\right] = 0 \quad . \qquad 2.4.8$$

Here, the various potentials which we have introduced can be defined by the fact that they satisfy a Poisson's equation of the form

$$\nabla^2\Psi = -4\pi G\rho_0$$
$$\nabla^2\Phi = -4\pi G\rho_0\varphi \quad , \qquad 2.4.9$$
$$\nabla^2\mathbf{Y} = -4\pi G\rho_0\mathbf{u}$$

where $\varphi = u^2 + \Psi + \frac{1}{2}\Pi + \frac{3}{2}(P/\rho_0)$.

Y is a vector potential whose source is the same as that of the Newtonian potential weighted by the local velocity field. Similarly, Φ is a scalar potential whose source is again that of the Newtonian potential but weighted by a function φ related to the total internal energy field.

Expansive as the equations of motion are we may still derive some comprehension for the meaning of the various terms in equation (2.4.8). The first two terms are basically Newtonian, indeed neglecting the contribution to the mass from energy $\sigma = \rho_o$ and they are identical to the first term of the Newtonian-Euler-Lagrange equations of hydrodynamic flow. The first part of the third term is just the pressure gradient and thus also to be expected on Newtonian grounds alone. The remaining contribution to the pressure gradient results from the space curvature introduced by the presence of the matter and is perhaps the most likely relativistic correction to be expected. The remaining tensors are the non-linear interaction terms alluded to earlier. The lengthy expression in braces contains the effects of the 'dragging' of the metric by the matter and the induced velocity dependent terms. The last term represents the direct effect of the matter-energy potentials on the metrics and this effect, in turn, is propagated to the potentials themselves.

Having obtained the equations of motion for the system, the procedure for obtaining the general relativistic form of Lagranges' identity is the same as we have used repeatedly in earlier sections. For simplicity, we shall compute the scalar version of Lagranges' identity by taking the inner product of equation (2.4.8) with the position vector **r** .We should expect this procedure to yield terms similar to the classical derivation but with differences introduced by differences between ρ_o , ρ^*, and σ. In addition we shall take our volumes large enough so that volume integrals of divergence vanish. In this regard it is worth noting that if volume contains the entire system, then by the divergence theorem

$$\int_V \nabla A dV = \tfrac{1}{3}\int_V \nabla \bullet (\mathbb{1}A)dV = 0 \ . \qquad 2.4.10$$

Thus, by integrating the equations of motion over the volume yields

$$\frac{d}{dt}\int_V \left\{\sigma\mathbf{u} + \frac{\rho_0}{c^2}\left[4\mathbf{u}\Psi - \frac{7}{2}\mathbf{Y} - \nabla\left(|\mathbf{Y}|\right)\right]\right\}dV = 0 \equiv \frac{d}{dt}\int_V \mathbf{K}\,dV \quad , \qquad 2.4.11$$

after noting that remaining integrals in the braces { }of equation (2.4.8) can be integrated by parts to zero. Equation (2.4.11) is a statement of conservation of linear momentum. This is a useful result for simplifying equation (2.4.8). Now we are prepared to write down Lagrange's identity by letting the integral of equation (2.4.11) be the local linear momentum density **K** and taking the scalar product of **r** with equation (2.4.8).

$$\begin{aligned}&\mathbf{r}\bullet\frac{d\mathbf{K}}{dt}-\rho_0\bullet\nabla\left[\Psi+P(1+2\Psi/c^2)\right]+\frac{4\rho}{c^2}\mathbf{r}\bullet[\mathbf{u}\bullet(\nabla\mathbf{Y})]-\tfrac{1}{2}\mathbf{r}\bullet(\mathbf{u}\bullet\nabla)\left[\mathbf{Y}-\nabla(|\mathbf{Y}|)\right]\\&\quad-\frac{2\rho}{c^2}\left(\varphi\mathbf{r}\bullet\nabla\Psi+\mathbf{r}\bullet\nabla\Phi\right)=0\end{aligned} \quad . \qquad 2.4.12$$

After multiple integration by parts and liberal use of the divergence theorem, this becomes

$$\frac{d}{dt}\int_V(\mathbf{r}\bullet\mathbf{K})dV=2T+\Omega+3\int_V\left(1+2\Psi/c^2\right)PdV+\frac{1}{c^2}\left[4\mathsf{W}+<\Phi>-\tfrac{7}{4}\mathsf{Y}-\tfrac{1}{4}\mathsf{Z}\right]\ , \qquad 2.4.13$$

where

$$\begin{aligned}T&=\tfrac{1}{2}\int_V\sigma u^2dV\\ \Omega&=-\tfrac{1}{2}\int_V\rho_0\Psi dV\\ \mathsf{W}&=\int_V\rho_0u^2\Psi dV\\ <\Phi>&=\int_V\rho_0\varphi\Psi dV\\ \mathsf{Y}&=\int_V\rho_0\mathbf{u}\bullet\mathbf{Y}dV\end{aligned} \quad , \qquad 2.4.14$$

and

$$\mathsf{Z}=\int_V\int_{V'}\left\{\rho_0\rho_0'\left(\left[\mathbf{u}\bullet(\mathbf{r}-\mathbf{r}')\right]\left[\mathbf{u}'\bullet(\mathbf{r}-\mathbf{r}')\right]\Big/|\mathbf{r}-\mathbf{r}'|^3\right)\right\}dV'dV\ .$$

This can be made somewhat more familiar if we re-write the left hand side of equation (2.4.13) so that

$$\begin{aligned}&\tfrac{1}{2}\frac{d^2}{dt^2}\left(\int_V\sigma r^2dV\right)+\frac{1}{c^2}\frac{d}{dt}\int_V\rho_0\left[4\Psi(\mathbf{r}\bullet\mathbf{u})-\tfrac{7}{2}\mathbf{r}\bullet\mathbf{Y}-r\bullet\nabla(|\mathbf{Y}|)\right]dV=2T+\Omega+3\int_V(1+2\Psi/c^2)PdV\\&\quad+\frac{1}{c^2}\left[4\mathsf{W}+<\Phi>-\tfrac{7}{2}\mathsf{Y}-\tfrac{1}{4}\mathsf{Z}\right]\end{aligned} \quad . \qquad 2.4.15$$

The first term on the left hand side of equation (2.4.15) is just $\tfrac{1}{2}d^2I/dt^2$ in the Newtonian limit. The second term arises from the correction to the metric resulting from the potential and the 'dragging' of the metric due to internal motion. The first two terms on the right are just what one would expect in the Newtonian limit while the next term can be related to the total internal energy.

This term contains a relativistic correction resulting directly from the change in metric due to the presence of matter. The remaining terms are all energy like and the first two (W and $<\Phi>$) represent relativistic corrections arising from the change in the potential caused by the metric modification by the potential itself. The last two involve metric dragging.

We have gone to some length to show the problems injected into the virial theorem by the non-linear aspects of general relativity. Writing Lagrange's identity as in equation (2.4.15) emphasizes the origin of the various terms - whether they are Newtonian or Relativistic. Although terms to this order should be sufficient to describe most phenomena in stellar astrophysics, we can ask if higher order terms or other metric theories of gravity provide any significant corrections to the virial theorem. The Einstein, Infield Hoffman approximation has been iterated up to 2 1/2 times[24, 25] in a true tour-de-force by Chandrasekhar and co-workers looking for additional effects. At the 2 1/2 level of approximation, radiation reaction terms appear which could be significant for non-spherical collapsed objects which exist over long periods of time. Using a parameterized version of the post-Newtonian approximation, Ni[26] has developed a set of hydrodynamic equations which must hold in nearly all metric theories of gravity and that depend on the values (near unity) of a set of dimensionless parameters. This latter effort is useful for relating various terms in the equations to the fundamental assumptions made by different theories.

Perhaps the most obvious lesson to be learned from the EIH approach to this problem is that continued application of the theory is not the way to approach the general results. However, of some consequence is the result that conservation laws for energy, momentum, and angular momentum exist and are subject to an operational interpretation at all levels of approximation. Thus it seems reasonable to conjecture that these laws as well as the virial theorem remained well posed in the general theory.

5. Complications: Magnetic Fields, Internal Energy, and Rotation

The full power and utility of the virial theorem does not really become apparent until one realizes that we need not be particularly specific about the exact nature of the potential and kinetic energies that appear in the earlier derivations. Thus the presence of complicating forces can be included insofar as they are derivable from a potential. Similarly as long as the total kinetic energy can be expressed in terms of energies arising from macroscopic motions and internal thermal motions, it will be no trouble to express the virial theorem in terms of these more familiar parameters of the system. One may proceed in just this manner or return to the original equations of motion for the system. We shall discuss both approaches.

In Chapter I, we derived the Euler-Lagrange equations of hydrodynamic flow. These equations of motion are completely general and are adequate to describe the effects of rotation and magnetic-fields if some care is taken with the coordinate frame and the 'pressure tensor' $\mathfrak{P}$. With this in mind, we may rewrite equation (1.1.4), noting that the left hand side is a total time derivative and that the pressure tensor can be explicitly split to include the presence of large scale electromagnetic fields. Thus

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla\Phi - \frac{1}{\rho}\nabla \bullet \left(\mathfrak{P}_g + \Im\right) - \frac{1}{\rho}\int \mathbf{S}(\mathbf{v}-\mathbf{u})dv \,, \qquad 2.5.1$$

where the tensor $\mathfrak{P}_g$ refers to the gas pressure alone and the tensor $\Im$ represents the Maxwell stress tensor for the electromagnetic field which has components[27]

$$\Im_{ij} = E_i D_j - \tfrac{1}{2}\delta_{ij}\sum_k E_k D_k + H_i B_j - \tfrac{1}{2}\delta_{ij}\sum_k H_k B_k \,, \qquad 2.5.2$$

or in dyadic notation

$$\Im = \mathbf{ED} - \tfrac{1}{2}\mathbb{1}[\mathbf{E}\bullet\mathbf{D}] - \mathbf{HB} - \tfrac{1}{2}\mathbb{1}[\mathbf{H}\bullet\mathbf{B}] \;. \qquad 2.5.3$$

For almost all cases in astrophysics, it is appropriate to ignore electrostriction and magnetostriction effects which complicate the relationships between **E** and **D** and **B** and **H**. In the absence of these body forces, the divergence of equation (2.5.3) yields[2.4]

$$\nabla \bullet \Im = \frac{1}{4\pi}\left\{\mathbf{D}\rho_e - \mathbf{D}\times(\nabla\times\mathbf{D}) + c^2\left[\mathbf{H}\times(\nabla\times\mathbf{H})\right]\right\} \;, \qquad 2.5.4$$

which is just the Lorentz force on the medium. It is useful when considering a configuration in uniform rotation to transform the problem into a co-rotating coordinate frame. This enables one to see the effect of macroscopic mass motion explicitly in the formalism and thus assess its interaction with other large scale properties of the system. In addition such systematic motion is represented by the stream velocity **u** in the collision term of the Boltzmann equation making its meaning clearer. Therefore, just as we have separated the effects of the electromagnetic field from the pressure tensor, let us explicitly represent the effects of rotation.

In transforming the inertial coordinate frame to a non-inertial rotating frame attached to the system, we must allow for temporal changes in any vector seen in one frame but not the other. Goldstein[28] gives a particularly lucid account of how this is to be accomplished by use of the operator

$$\left(\frac{d}{dt}\right)_{\text{inertial}} = \left(\frac{d}{dt}\right)_{\text{non-inertial}} + \boldsymbol{w}\times \quad , \qquad 2.5.5$$

where $\boldsymbol{w}$ is the angular velocity appropriate for the point function upon which the operator acts (*i.e. the angular velocity of the rotating frame*).

If we let

$$\mathbf{u} = \mathbf{w} + (\boldsymbol{w}\times\mathbf{r}) \;, \qquad 2.5.6$$

then the equations of motion for uniform rotation (i.e., $\boldsymbol{w}$ = constant) become

$$\frac{d\mathbf{w}}{dt} + 2(\boldsymbol{w}\times\mathbf{w}) + \boldsymbol{w}\times(\boldsymbol{w}\times\mathbf{r}) = -\nabla\Phi - \frac{1}{\rho}[\nabla\bullet(\mathfrak{P}+\Im)] - \frac{1}{\rho}\int_V \mathbf{S}(\boldsymbol{w})d\mathbf{v} \;. \qquad 2.5.7$$

It can be shown that for uniform rotation the term[2.5].

$$\boldsymbol{w}\times(\boldsymbol{w}\times\mathbf{r}) = \tfrac{1}{2}\nabla[(\boldsymbol{w}\times\mathbf{r})\bullet(\boldsymbol{w}\times\mathbf{r})] \quad , \qquad 2.5.8$$

so that it may be combined with the gravitational potential and $\frac{1}{2}(\boldsymbol{w}\times\mathbf{r})^2$ may be considered a 'rotational potential'. The term $2(\boldsymbol{w}\times\mathbf{w})$ is known as the "coriolis force". Since both the pressure tensor and Maxwell tensor are normally "fixed" to the body, we should expect their formulation in the rotating frame to be simpler. It is difficult to proceed much further with the equations of motion without making some simplifying assumptions. The most helpful and also reasonable of these is to assume that all collisions processes are isotropic in space. This has two results. Firstly, the collision term on the right hand side of the equation of motion averages to zero when integrated over all velocity space. Secondly, the gas pressure tensor $\mathfrak{P}_g$ becomes diagonal with all elements equal and can thus $\nabla\bullet\mathfrak{P}_g = \mathrm{P}$, where P is the scalar gas pressure. This effectively guarantees the existence of a scalar equation of state which will be useful later when relating P to the internal energy. Since almost all astrophysical situations relate to plasmas, we may consider the configurations to have a very large conductivity and can therefore neglect the contribution to the Maxwell Field tensor of electric fields. Using these assumptions, and equations (2.5.4) and (2.5.8), the equation of motion becomes

$$\frac{d\mathbf{w}}{dt}+2(\boldsymbol{w}\times\mathbf{w}) = -\nabla\left[\Phi+(\boldsymbol{w}\times\mathbf{r})^2\right]-\nabla \mathrm{P}/\rho-\frac{1}{4\pi\rho}\left[\mathbf{H}\times(\nabla\times\mathbf{H})\right] \quad . \qquad 2.5.9$$

As we have done before we shall multiply the equations of motion by $\rho\mathbf{r}$, and integrate over the volume and generate a general tensor expression for Lagrange's identity including rotation and magnetic fields.[2.6] Thus,

$$\tfrac{1}{2}\frac{d^2}{dt^2}(\mathfrak{I})+2\boldsymbol{w}\bullet\mathfrak{L} = 2\mathfrak{T}+n\mathfrak{U}+\boldsymbol{w}\bullet(\mathbb{1}\boldsymbol{w})-\omega^2\mathbb{1}\cdot\int_V \mathbf{r}\nabla \mathrm{P}dV-\frac{1}{4\pi}\int_V \mathbf{r}[\mathbf{H}\times(\nabla\times\mathbf{H})]dV \quad . \qquad 2.5.10$$

where $\mathfrak{I}$, $\mathfrak{T}$, and $\mathfrak{U}$ are moment of inertia, kinetic and potential energy tensors respectively. $\mathfrak{L}$ is an angular momentum-like tensor[2.6]. In order to simplify the last two terms it is worth noting that they are derived from the divergence of two tensors. So, we may use the chain rule followed by the divergence theorem to simplify these terms. Thus

$$\int_V \mathbf{r}\nabla\bullet(\mathfrak{P}+\Im)dV = \int_V \nabla\bullet[\mathbf{r}(\mathfrak{P}+\Im)dV-\int_V(\nabla\mathbf{r})\bullet(\mathfrak{P}+\Im)dV \quad . \qquad 2.5.11$$

The divergence theorem guarantees that the first integral can be written as a surface integral and if the volume is taken to be large enough can always be made to be zero. However, since magnetic fields always extend beyond the surface of what one normally considers the surface of the configuration, we will keep these terms for the moment. Thus

$$\int_V \nabla\bullet[\mathbf{r}(\mathfrak{P}+\Im)dV = \int_S \mathbf{r}(\mathfrak{P}+\Im)\bullet d\mathbf{S} \equiv \mathfrak{S}, \qquad 2.5.12$$

or, in terms of the components, the surface terms can be written as

$$\mathfrak{S}_{ij} = \frac{1}{4\pi}\int_S x_j(H_i\sum_k H_k dS_k - \frac{1}{8\pi}\int_S x_j H^2 dS_i - \int_S P_0 x_j dS_i \quad . \qquad 2.5.12a$$

Keeping in mind that $\nabla\mathbf{r} = \mathbb{1}$ (i.e., a second rank tensor with components δ_{ij}), the second integral in equation (2.5.11) becomes

$$\int_V \mathbb{1} \bullet (\mathfrak{P} + \mathfrak{I}) dV = \int_V \mathbb{1}(P + H^2/8\pi) dV + \frac{1}{4\pi}\int_V (\mathbf{HH}) dV \quad . \qquad 2.5.13$$

Defining

$$\mathfrak{M} \equiv \frac{1}{8\pi}\int_V (\mathbf{HH}) dV \quad , \qquad 2.5.14$$

we arrive at the final tensor form for Lagrange's Identity, including rotation and magnetic fields.

$$\tfrac{1}{2}\frac{d^2\mathfrak{J}}{dt^2} + 2\boldsymbol{w} \bullet \mathfrak{L} = 2[\mathfrak{T} - \mathfrak{M}] + n\mathfrak{U} + \boldsymbol{w} \bullet (\mathbb{1}\boldsymbol{w}) - \omega^2 \mathbb{1} + \mathbb{1}\left(\int_V (P + H^2/8\pi)\, dV\right) + \mathfrak{S} , \qquad 2.5.15$$

where $\mathfrak{L}_{ijk} = \int_V \rho\left(r_k r_i \varpi_j - \varpi_k r_i r_j\right) dV$.

We have suffered through the tensor derivation in order to show the complete generality of this formalism. The tensor component equations are essential for investigating non-radial oscillations and other such phenomena which cannot be represented by a simple scalar approach. However, it is easier to appreciate the physical significance of this approach by looking at the scalar counterpart of equation (2.5.15). In section 1, we pointed out that the scalar form is derived by taking "inner" products of the position vector with the equation of motion while the tensor virial theorem involves "outer" or tensor products. One may either repeat the derivation of equation (2.5.15) taking "inner" products or contract, the component form of equation (2.5.15) over indices i and j.

The contraction of the tensors $\mathfrak{J}$, $\mathfrak{T}$, $\mathfrak{M}$, $\mathfrak{U}$, and $\mathbb{1}$ yield the moment of inertia about the origin of the coordinate system I, the kinetic energy due to internal motion T, the total magnetic energy $\mathcal{M}$, the total potential energy Φ respectively. Some care must be taken in contracting the tensors $\mathfrak{L}$, and $\mathbb{1}\boldsymbol{w}$. From the definition of $\mathfrak{L}$, in equation (N2.6.3), it is clear that the contracted form of that expression can be written as

$$2\int_V \rho\mathbf{r} \bullet (\boldsymbol{w} \times \mathbf{w}) dV = -2\boldsymbol{w} \bullet \int_V \rho(\mathbf{r} \times \mathbf{w}) dV = 2\boldsymbol{w} \bullet \int_V \boldsymbol{l} dV , \qquad 2.5.16$$

where $\boldsymbol{l}$ is the net volume angular momentum density on the material due to coriolis forces. We can again choose our rotating frame so that $\int \boldsymbol{l} dV$ is zero and this term must vanish from the contracted equation. The simplest method for deriving the value of the contracted form of $\boldsymbol{w} \bullet (\mathbb{1}\boldsymbol{w} - \boldsymbol{w}\mathbb{1})$ in equation (2.5.15) is again to examine the contracted form of the term giving rise to it. Since $\mathbf{w} = w \mathbf{\ x\ r}$, we can expand the left hand side of equation (2.5.16) by means of identities relating to the vector triple product [see note *2.6*, specifically equation (N2.6.4)], and obtain

$$\int_V \rho\mathbf{r} \bullet [\boldsymbol{w} \times (\boldsymbol{w} \times \mathbf{r})] dV = \int_V \rho\mathbf{r} \bullet [\boldsymbol{w} \times \mathbf{v}] dV = \int_V \rho[\mathbf{r} \times \boldsymbol{w}] \bullet \mathbf{v} dV = \int_V \rho v^2 dV = 2\mathfrak{R} , \qquad 2.5.17$$

where $\mathfrak{R}$ is just the total energy due to rotation. Substitution of the contraction of these tensors into equation (2.5.15) yields a much simpler result

$$\tfrac{1}{2}\frac{d^2I}{dt^2} = 2(T+\Re) + \mathcal{M} - \Omega + n\mathcal{U} + 3\int_V PdV + \sum_i \mathfrak{S}_{ii} \quad . \qquad 2.5.18$$

The contraction of the surface terms yield the scalar integrals

$$\sum_i \mathfrak{S}_{ii} = \frac{1}{4\pi}\int_S (\mathbf{r}_0 \bullet \mathbf{H}_0)(\mathbf{H}_0 \bullet d\mathbf{S}) - \int_S \left(P_0 + H_0^2/8\pi\right)\mathbf{r} \bullet d\mathbf{S} \, . \qquad 2.5.19$$

Here, the subscript "0" indicates the value of the variables on the surface. In practice these integrals are usually small compared with the magnitude of the volume integrals found in the remainder of the expression. So far, we have said little about the contribution of the volume integral of the pressure. Since even in the tensor representation this term appears as a scalar, there was no loss of generality in deferring the evaluation of the integral until now. Dimensional analysis will lead one to the result that the pressure integral is an 'energy-like' integral. From thermodynamic consideration, we can write the internal "heat energy" $\mathcal{U}$ as

$$\mathcal{U} = \int_V \rho c_v \mathcal{T} dV \, , \qquad 2.5.20$$

where c_v is the specific heat of constant volume and is the temperature. We also know that the kinetic energy associated with the material is

$$\mathcal{E} = \tfrac{3}{2}Nk\mathcal{T} = \tfrac{3}{2}\rho\left(c_p - c_v\right)\mathcal{T} - \tfrac{3}{2}P \quad . \qquad 2.5.21$$

Combining these two equations we get the total internal "heat energy" as

$$\mathcal{U} = \tfrac{2}{3}\int_V \frac{\mathcal{E}dV}{\left(c_p/c_v - 1\right)} = \int_V \frac{\rho dV}{\left(c_p/c_v - 1\right)} \quad . \qquad 2.5.22$$

It is traditional to let $\gamma = c_p/c_v$ and if we let this be constant throughout the volume and neglect the surface term in equation (2.5.18), then we can write the scalar form of Lagrange's identity as

$$\tfrac{1}{2}\frac{d^2I}{dt^2} = 2(T+\Re) + \mathcal{M} - \Omega + 3(\gamma-1)\mathcal{U} \quad . \qquad 2.5.23$$

At the beginning of this discussion, I said that one could either derive Lagrange's identity from the equations of motion or from careful consideration of meanings of potential and kinetic energy. For example, the first and last terms on the right hand side of equation (2.5.23) are just the contribution to the total kinetic energy of the system from macroscopic motions, rotation and thermal motion respectively. The remaining terms are just a specification of the nature of the total potential energy. Thus, by realizing that $3(\gamma\text{-}l)\mathcal{U}$ is just twice the kinetic energy due to thermal motion we could have written equation (2.5.23) down immediately. However, it is unlikely that this could have been done for equation (2.5.15). Since the variational form of this equation will be useful in the next chapter, our efforts have not been wasted.

Before leaving this section on complicating phenomena, there is one last aspect to be investigated. In Chapter I, section 3, we noted that the inclusion of velocity dependent forces such as frictional force do not alter the results of the virial theorem as they can be averaged to zero given sufficient time. This result was apparently first noted by E. H. Milne[29] in 1925. Although no modification is made to the virial theorem some effects can be seen in Lagrange's identity and so, let us take a moment to recapitulate these arguments. If we have a volume force which can be derived from a "friction" tensor, then one would add terms to the equations of motion of the form

$$\boldsymbol{f}_{\mathrm{f}} = \rho \mathbf{w} \bullet \mathbf{f} \,, \qquad 2.5.24$$

which make the following contribution to the tensor form of Lagrange's identity:

$$\int_V \mathbf{r}\boldsymbol{f}_{\mathrm{f}} \mathrm{dV} = \int_V \rho \mathbf{r}\mathbf{w} \bullet \mathbf{f}\mathrm{dV} = \tfrac{1}{2}\int_V \rho\left[\frac{\mathrm{d}}{\mathrm{dr}}(\mathbf{rr})\right] \bullet \mathbf{f}\mathrm{dV} \;. \qquad 2.5.25$$

If **f** were indeed constant throughout the configuration, the right-hand side of equation (2.5.25) would just become

$$\tfrac{1}{2}\left(\frac{\mathrm{d}\mathfrak{I}}{\mathrm{dt}}\right) : \mathbf{f} \,, \qquad 2.5.26$$

and Lagrange's identity in its full generality would be

$$\tfrac{1}{2}\frac{\mathrm{d}^2\mathfrak{I}}{\mathrm{dt}^2} + \tfrac{1}{2}\left(\frac{\mathrm{d}\mathfrak{I}}{\mathrm{dt}}\right) : \mathbf{f} + 2\boldsymbol{w} \bullet \mathfrak{L} = 2(\mathfrak{T} - \mathfrak{M}) + \mathrm{n}\mathfrak{U} + \boldsymbol{w} \bullet [\boldsymbol{l}\boldsymbol{w} - \boldsymbol{w}\boldsymbol{l}] + \boldsymbol{l}[(1-\gamma)\mathcal{U} + \mathcal{M}] + \mathfrak{S} \;. \qquad 2.5.27$$

6. Summary

In this chapter we have continued the development of the virial theorem as it appears in more contemporary usage. The tensor virial theorem is a more general form of Lagrange's identity which when averaged over time provides rather general expressions for the coordinated behavior of some energy like tensors of the system. Further insight into the nature of this process is discovered in the second section where we find that taking higher order spatial moments of the equation of motion is equivalent to recovering the information selectively lost in the classical derivation of the virial theorem. In principle this approach could be used in a prescription for the complete solution of the equations of motion. However, it seems likely that in practice it would be more difficult than implementing a direct numerical solution of the original equations themselves. The importance of the method lies in the fact that such moment expressions for stable systems are normally rapidly convergent. Thus, the largest amount of information can be recovered with the least effort.

In the next two sections, we considered the effects of a relativity principle on the development of the virial theorem. We found that large velocities require large gravitational fields to keep them in check and thus one might argue that a separate discussion of the effects of special relativity is not warranted. However, there are at least two dynamically stable systems for which this is not true (i.e., pure radiation spheres which approximate some models of super massive stars and white dwarfs where low-mass, high velocity electrons, were kept in check by the high-mass, low velocity nucleons). In addition, Lagrange's identity is applicable to systems which are not in equilibrium and hence may be relativistic. For this reason, we have developed the special and general relativistic versions of the theorem separately and will return in the last chapter to discuss some specific applications of them. I have attempted throughout the chapter to emphasize the similarity of the derivation of the virial theorem and particularly in section 3, the perceptive reader may have noticed that the derivation is equivalent to carrying out

$$\int_V \mathbf{r} \bullet \Box \bullet \Im \, dV = 0 , \tag{2.6.1}$$

where **r** is a four vector in the Lorentz metric. However, we ignored all contributions from the time-like part. These would have been of the form

$$\tfrac{1}{2}\frac{d}{dt}\int_V t\Im_{00} dV = \int_V \left(t\sum_j \frac{\partial \Im_{0j}}{\partial x_j} \right) dV , \tag{2.6.2}$$

which after appropriate application of the divergence theorem, becomes

$$\frac{d}{dt}\left(\int_V \varepsilon dV \right) = \frac{dE}{dt} = -\int_S \frac{\rho c^2}{\gamma} \frac{d\mathbf{x}}{dt} \bullet d\mathbf{S} . \tag{2.6.3}$$

Because of the linear independence of the time and space coordinates, this is not a new result but rather an expression of the conservation of energy. Essentially it states that the time rate of change of the total energy of the system equals the momentum flux across the system boundary multiplied by c^2. When the metric is no longer the 'flat' Lorentz metric as in the case in section 4, things are no longer simple. It is this loss of simplicity which caused me to stray from the more rigorous approaches of other sections. Thus rather than stress the manipulative complexity of the post-Newtonian approximation, I have attempted to provide physical motivation from the existence of terms arising in the equations of motion that results from the non-linear nature of general relativity. The derivation presented in this section follows exactly the prescription of earlier sections, but for simplicity I presented the development of the scalar version of the virial theorem only. Sticking with the post-Newtonian approximation avoided some difficult problems of uniqueness and interpretation.

However, I remain convinced that a general formulation of Lagrange's identity and the virial theorem which is compatible with the field equation of Einstein exists and its formulation would be most rewarding[†].

†____

Since the Pachart edition of this effort was written, there has been some significant progress in this area. Bonazzola[30]*, formulated the virial theorem for the spherical, stationary case in full general relativity. He notes that in the non-stationary, non-spherical case, the existence of gravitational radiation destroys any strict equivalence between the general relativistic and Newtonian cases so that there can be no unique formulation of the virial theorem. Thus any formulation of a virial-like relation will depend on the specific nature of the configuration. This approach was extended by Vilain*[31] *who applied a similar formulation of a general relativistic virial theorem to the stability of perfect fluid spheres.*

For most stellar astrophysical applications the post-Newtonian result is probably sufficient. In the last section of this chapter some of the powers of the virial theorem to deal with difficult situations became apparent. The results which have been generalized to include additional effects are not the result of any new physical concepts. Rather, they are the result of the specific identification of the physical contributions to the system made by such attributes as magnetic fields and macroscopic motion. Although I included only magnetic fields throughout most of the discussion, the inclusion of electric fields in the Maxwell field tensor makes it clear how to proceed should they be present. Lastly we looked again at velocity dependent forces not so much with an eye to their effect on the virial theorem, but rather with a view to their persistent presence in the variational form of Lagrange's identity. The presence of all these complicating aspects is included only to make their interplay explicit. The basic theorem must hold; all the rest is done to glean more insight.

Notes to Chapter 2

2.1 The term on the left of equation (2.1.2) becomes

$$\int_V \rho\mathbf{r}\frac{d\mathbf{u}}{dt}dV = \int_V \rho\mathbf{r}\frac{d^2\mathbf{r}}{dt^2}dV = \int_V \rho\mathbf{r}\frac{d}{dt}\left(\frac{\mathbf{r}d\mathbf{r}}{dt}\right)dV - \int_V \rho\frac{d\mathbf{r}}{dt}\frac{d\mathbf{r}}{dt}dV \quad . \qquad \text{N2.1.1}$$

The third term can further be simplified So that

$$\int_V \rho\mathbf{r}\frac{d}{dt}\left(\frac{\mathbf{r}d\mathbf{r}}{dt}\right)dV = \tfrac{1}{2}\int_V \rho\frac{d^2}{dt^2}(\mathbf{rr})dV = \tfrac{1}{2}\frac{d^2}{dt^2}\left(\int_V \rho\mathbf{rr}\right)dV \quad . \qquad \text{N2.1.2}$$

In obtaining the third term in equation (2.1.4) we have assumed that the volume V is large enough to contain all matter and the conservation of mass argument explicitly developed in Chapter I has the form

$$\tfrac{1}{2}\frac{d^2}{dt^2}\left(\int_V \rho\mathbf{rr}\right)dV = \int_V (\rho\mathbf{uu})dV + \int_V \rho\mathbf{r}\nabla\Phi dV \,. \qquad \text{N2.1.3}$$

2.2 If one writes equation (2.1.6) in component form, one gets

$$\tfrac{1}{2}\frac{d^2\mathfrak{I}_{ij}}{dt^2} = 2\mathfrak{T}_{ij} + n\mathfrak{U}_{ij} \quad , \qquad \text{N2.2.1}$$

where in Cartesian Coordinates the tensor components are

$$\begin{aligned}
\mathfrak{I}_{ij} &= \int_V \rho x_i x_j dV \\
\mathfrak{T}_{ij} &= \int_V \rho u_i u_j dV \\
\mathfrak{U}_{ij} &= \tfrac{1}{2}\int_V\int_{V'} \rho(\mathbf{r})\rho(\mathbf{r}')(x_i - x'_i)(x_j - x'_j)\left(|\mathbf{r}-\mathbf{r}'|\right)^{n-2} dV'dV
\end{aligned} \quad . \qquad \text{N2.2.2}$$

By contracting these tensors and letting the potential be that of the gravitational field, we recover the scalar form of Lagrange's identity as given in Chapter I, equation (1.4.12).

2.3 Taking the scalar product of a space-like position vector with equation (2.3.6) and integrating over a volume sufficient to contain the system we get

$$c\int_V \sum_j x_j \frac{\partial(\rho u_j)}{\partial t}dV = \sum_j\sum_k\int_V x_j\frac{\partial \Im_{jk}}{\partial x_k}dV = 0 \quad . \qquad \text{N2.3.1}$$

From the chain rule the second integral can be written as

$$\sum_j\sum_k\int_V x_j\frac{\partial \Im_{jk}}{\partial x_k}dV = \sum_j\sum_k\int_V \frac{\partial}{\partial x_k}\left(x_j\Im_{jk}\right)dV - \sum_j\sum_k\int_V \frac{\partial x_j}{\partial x_k}\Im_{jk}dV \,. \qquad \text{N2.3.2}$$

The first integral on the right is just the integral of the divergence of $x_j\Im_{jk}$ over V and if the volume is chosen to enclose the entire system the integral must vanish as $\Im_{jk} = 0$ on the surface enclosing V (i.e., Gauss's Law applies). Since $\partial x_i/\partial x_j = \delta_{ij}$, the second integral becomes $\sum_j \int_V \Im_{jj} dV$. Equation (N2.3.1) is then

$$\sum_j c\int_V x_j \frac{\partial}{\partial t}(\rho u_j)dV - \sum_j \int_V \Im_{jj} dV = 0 \quad . \qquad \text{N2.3.3}$$

Now, $\left(\partial x_j \big/ \partial t\right) = 0$ from the orthogonality of the Lorentz frame and $\sum_{j=1}^{3} \Im_{jj} = \sum_{\alpha=0}^{3} \Im_{\alpha\alpha} - \Im_{00}$, so we can write

$$c\int_V \frac{\partial}{\partial t}\left(\sum_j \rho x_j u_j\right)dV + \int_V \rho c^2 \gamma dV + \int_V \Im_{00} dV = 0 \quad . \qquad \text{N2.3.4}$$

With the sign convection $\Im_{00}$ is the negative of the total energy density, Bergmann[17], among others, shows us that the "relativistic" kinetic energy density τ is given by

$$\tau = \rho c^2(\gamma^{-1} - 1)$$

or

$$\gamma\tau = \rho c^2 - \rho c^2 \gamma \quad . \qquad \text{N2.3.5}$$

Thus, using equation (2.3.3) to re-write the first term of equation (N2.3.4), we get

$$\sum_{j=1}^{3}\int_V \frac{\partial}{\partial t}\left[\tfrac{1}{2}\rho \frac{d(x_j x_j/\gamma)}{dt}\right]dV + \int_V (\rho c^2 - \gamma\tau - \varepsilon - \rho c^2)dV = 0 \,, \qquad \text{N2.3.6}$$

where ε is the potential energy density. Applying Leibniz's law for the differentiation of definite integrals[18] to the first term in equation (N2.3.6) and re-writing the second one, we get

$$\sum_{j=1}^{3}\int_V \frac{\partial}{\partial t}\left[\tfrac{1}{2}\rho \frac{d(x_j x_j/\gamma)}{dt}\right]dV = \Omega + T + \int_V \gamma\tau \, dV \,. \qquad \text{N2.3.7}$$

2.4

$$\Im = \frac{1}{4\pi}\left[\mathbf{DD} + c^2\mathbf{HH} - \tfrac{1}{2}\mathbb{1}(D^2 + c^2H^2)\right] \quad . \qquad \text{N2.4.1}$$

If we take the divergence of $\Im$ we get

$$\nabla \bullet \Im = \frac{1}{4\pi}\left\{(\mathbf{D}\bullet\nabla)\mathbf{D} + \mathbf{D}(\nabla\bullet\mathbf{D}) + (c^2\mathbf{H}\bullet\nabla)\mathbf{H} + c^2\mathbf{H}(\nabla\bullet\mathbf{H}) - \tfrac{1}{2}\nabla(\mathbf{D}\bullet\mathbf{D} + c^2\mathbf{H}\bullet\mathbf{H})\right\}, \qquad \text{N2.4.2}$$

and invoking Maxwell's laws that $\nabla \bullet \mathbf{D} = \rho_e$ and $\nabla \bullet \mathbf{H} = 0$, this becomes

$$\nabla \bullet \Im = \frac{1}{4\pi}\left\{(\mathbf{D}\bullet\nabla)\mathbf{D} + \mathbf{D}\rho_e + (c^2\mathbf{H}\bullet\nabla)\mathbf{H} - \tfrac{1}{2}\nabla(\mathbf{D}\bullet\mathbf{D} + c^2\mathbf{H}\bullet\mathbf{H})\right\}. \qquad \text{N2.4.3}$$

Making use of the vector identity

$$\nabla(\mathbf{A}\bullet\mathbf{G}) = \mathbf{A}\times(\nabla\times\mathbf{G}) + (\mathbf{A}\bullet\nabla)\mathbf{G} + \mathbf{G}\times(\nabla\times\mathbf{A}) + (\mathbf{G}\bullet\nabla)\mathbf{A} \quad , \qquad \text{N2.4.4}$$

equation (N2.4.3) takes on the more familiar form

$$\nabla \bullet \Im = \frac{1}{4\pi}\left\{(\mathbf{D}\rho_e - \mathbf{D}\times(\nabla\times\mathbf{D}) + c^2[\mathbf{H}\times(\nabla\times\mathbf{H})]\right\}. \qquad \text{N2.4.5}$$

2.5 By use of the vector identity equation (N2.4.4), it is clear that

$$\tfrac{1}{2}\nabla(\boldsymbol{w}\times\mathbf{r})^2 = (\boldsymbol{w}\times\mathbf{r})\times[\nabla\times(\boldsymbol{w}\times\mathbf{r})] + [(\boldsymbol{w}\times\mathbf{r})\bullet\nabla](\boldsymbol{w}\times\mathbf{r}) \ , \qquad \text{N2.5.1}$$

but

$$\nabla\times(\boldsymbol{w}\times\mathbf{r}) = (\boldsymbol{w}\bullet\nabla)\mathbf{r} - (\mathbf{r}\bullet\nabla)\boldsymbol{w} - \boldsymbol{w}(\nabla\bullet\mathbf{r}) + \mathbf{r}(\nabla\bullet\boldsymbol{w})\,. \qquad \text{N2.5.2}$$

The constancy of $\boldsymbol{w}$ causes the second and fourth term to vanish, while the first third terms are equal but of opposite sign. Again, since $\boldsymbol{w}$ is constant.

$$[(\boldsymbol{w}\times\mathbf{r})\bullet\nabla](\boldsymbol{w}\times\mathbf{r}) = \boldsymbol{w}\times[(\boldsymbol{w}\times\mathbf{r})\bullet\nabla](\mathbf{r})\,. \qquad \text{N2.5.3}$$

In component form

$$[(\boldsymbol{w}\times\mathbf{r})\bullet\nabla](\mathbf{r}) = \sum_i\sum_j\sum_k \mathfrak{e}_{ijk}\omega_j r_k \frac{\partial x_\ell}{\partial x_i} = \sum_j\sum_k \mathfrak{e}_{\ell jk}\omega_j r_k = (\boldsymbol{w}\times\mathbf{r}) \ , \qquad \text{N2.5.4}$$

where $\mathfrak{e}_{ijk}$ in a completely antisymmetric tensor of rank 3 sometimes called the Levi Civita tensor density. Thus

$$\tfrac{1}{2}\nabla(\boldsymbol{w}\times\mathbf{r})^2 = \boldsymbol{w}\times(\boldsymbol{w}\times\mathbf{r}). \qquad \text{N2.5.5}$$

2.6

$$\int_V \rho\mathbf{r}\frac{d\mathbf{w}}{dt}dV + 2\int_V \rho\mathbf{r}(\boldsymbol{w}\times\mathbf{w})dV = -\int_V \rho\nabla\left[\Phi + (\boldsymbol{w}\times\mathbf{r})^2 + P/\rho\right]dV - \frac{1}{4\pi}\int_V \mathbf{r}[\mathbf{H}\times(\nabla\times\mathbf{H})]dV\,. \qquad \text{N2.6.1}$$

As in section 3, the first term on the left becomes

$$\int_V \rho\mathbf{r}\frac{d\mathbf{w}}{dt}dV = \tfrac{1}{2}\frac{d^2}{dt^2}\left(\int_V \rho(\mathbf{rr})dV\right) - \int_V \rho(\mathbf{ww})dV\,. \qquad \text{N2.6.2}$$

The second term is more difficult to simplify. Let the local velocity field $\mathbf{w} = \boldsymbol{\varpi}\times\mathbf{r}$ by defining a <u>*local*</u> angular velocity field $\boldsymbol{\varpi}$. Then, by expanding the resulting vector triple product in the second term we can write

$$2\int_V \rho\mathbf{r}(\boldsymbol{w}\times\mathbf{w})dV = 2\int_V \rho\mathbf{r}[\boldsymbol{w}\times(\boldsymbol{\varpi}\times\mathbf{r})]dV = 2\int_V \rho\mathbf{r}[\boldsymbol{\varpi}(\boldsymbol{w}\bullet\mathbf{r}) - \mathbf{r}(\boldsymbol{w}\bullet\boldsymbol{\varpi})]dV \qquad \text{N2.6.3}$$

or

$$2\int_V \rho\mathbf{r}(\boldsymbol{w}\times\mathbf{w})dV = 2\boldsymbol{w}\bullet\int_V \rho[\mathbf{r}(\mathbf{r}\boldsymbol{\varpi}) - \boldsymbol{\varpi}(\mathbf{rr})]dV = 2\boldsymbol{w}\mathfrak{L}\,,$$

where the tensor $\mathfrak{L}$ is an angular moment-like three index tensor representing the various components of volume net angular momentum within the body. Thus, $\boldsymbol{w}\bullet\mathfrak{L}$ is a kinetic energy like tensor resulting from the net motions induced by the coriolis forces. However, since there can be no net motions about any axis other than that defined by the total angular momentum of the body all components of $\boldsymbol{w}\bullet\mathfrak{L}$ must be zero except those associated with the axis of rotation. In addition we can choose our rotating frame $\boldsymbol{w}$ so that in that frame the net angular momentum is zero and all

contributions from the term $\boldsymbol{w}\bullet\mathfrak{L}$ will vanish. For the sake of generality we shall keep the term for the present. The first term on the right of equation (N2.6.1) is given by equation (2.1.4). Since we just went to some length to show that, $\frac{1}{2}\nabla(\boldsymbol{w}\times\mathbf{r})^2 = \boldsymbol{w}\times(\boldsymbol{w}\times\mathbf{r})$, we shall use the earlier version and an expansion of the vector triple product to evaluate the second term on the right.
Thus

$$\tfrac{1}{2}\int_V \rho\mathbf{r}\nabla(\boldsymbol{w}\times\mathbf{r})^2 dV = \int_V \rho\mathbf{r}[\boldsymbol{w}\times(\boldsymbol{w}\times\mathbf{r})]dV = \int_V \rho\mathbf{r}\boldsymbol{w}(\boldsymbol{w}\bullet\mathbf{r})dV - \int_V \rho\mathbf{r}\mathbf{r}\omega^2 dV\ , \qquad \text{N2.6.4}$$

or
$$\tfrac{1}{2}\int_V \rho\mathbf{r}\nabla(\boldsymbol{w}\times\mathbf{r})^2 dV = \boldsymbol{w}\bullet\left(\int_V \rho\mathbf{r}\mathbf{r}\boldsymbol{w}dV\right) - \omega^2\int_V \rho\mathbf{r}\mathbf{r}dV\ .$$

It is worth noting at this point that all terms dealt with so far involve only the volume integrals $\int_V \rho\mathbf{r}\mathbf{r}dV$ and $\int_V \rho\mathbf{w}\mathbf{w}dV$ which are the same as the tensors defined in Section 1. Thus by combining equations (N2.6.2), (N2.6.3), (N2.6.4), and (2.1.4), we may assess our progress in simplifying equation (N2.6.1) so far [see equation (2.5.10)].

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