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III The Variational Form of the Virial Theorem

1. Variations and Perturbations and their Implications for the Virial Theorem

Perturbation analysis is truly an old mechanism in which one explores the behavior of a system in a known state by assuming there are small variations of the independent variables describing the system and determining the individual variation in the independent variables. The vehicle for determining the independent variable changes is found in the very equations which describe the initial state of the system. The equations usually chosen for this type of analysis are the equations of motion for the system. For example, consider the equations of motion for an object moving under the influence of a point potential Φ.

$$\frac{d^2\mathbf{r}}{dt^2} = -\nabla\Phi \,. \qquad 3.1.1$$

Assume a solution $\mathbf{r}_o(t)$ is known, which satisfies equation (3.1.1) for a particular potential Φ_o. Since equation (3.1.1) is valid for any system where Φ is known, one could define

$$\left.\begin{array}{l} \Phi = \Phi_0 + \delta\Phi \\ \mathbf{r} = \mathbf{r}_0 + \delta\mathbf{r} \end{array}\right\} \,, \qquad 3.1.2$$

and equation (3.1.1) would require

$$\frac{d^2(\mathbf{r} + \delta\mathbf{r})}{dt^2} = -\nabla(\Phi + \delta\Phi)\,. \qquad 3.1.3$$

However, since both ∇ and d/dt are linear operators, equation (3.1.3) becomes

$$\frac{d^2\mathbf{r}_0}{dt^2} + \frac{d^2\delta\mathbf{r}}{dt^2} = -\nabla\Phi - \nabla\delta\Phi \,, \qquad 3.1.5$$

but we already know that

$$\frac{d^2\mathbf{r}_0}{dr^2} = -\nabla\Phi_0 \,, \qquad 3.1.6$$

so that by subtracting equation (3.1.5) from equation (3.1.4), we get

$$\frac{d^2\delta\mathbf{r}}{dr^2} = -\nabla\delta\Phi \,, \qquad 3.1.7$$

which we called the perturbed equations of motions where $\delta\Phi$ is the perturbation that involves the perturbation $\delta\mathbf{r}$. A short approach which leads to the same result is to "take the variation" of equation (3.1.1) wherein the operator δ is not affected by time or space derivatives. This technique "works" because the time and space operators in the equation of motion are linear, hence any linear perturbation or departure from a given solution will produce the sum of the original equations of motion on the perturbed equations of motion. In general, I shall use the variational operator δ in this sense, that is, it represents a small departure of a variable from the value it had which satisfies the equations governing the system. It is not necessary that one perturbs the equations of motion in order to gain information about the system. Clearly any equations which describe the structure of the system are subject to this type of analysis. Thus, if taking variations of the equations of motion produces useful results, might not the variational form of the moments of those equations also be expected to contain interesting information? It was in this spirit that Paul Ledoux developed the variational form of the scalar virial theorem[1], and was able to predict the pulsational period of a star.

The variational approach yields differential equations which describe parameter relationships for a system disturbed from an initial state. If that state happens to be an equilibrium state, variational analysis of the equations of motion would yield a description of the system motion about the equilibrium configuration. Variational analysis of spatial moments could then be expected to yield macroscopic properties of that motion. This is indeed the case, as Ledoux[1] demonstrated by determining perhaps the most obvious macroscopic property of such motion, the pulsational period of the system. Chandrasekhar[2] found the tensor form of the virial theorem useful in determining non-radial modes of oscillation of stars. In addition, he and Fermi[3] investigated the effects of a magnetic field on the pulsation of a star. An additional macroscopic property closely connected with the pulsational period, with which this approach deals, is the global stability of the system. We shall examine this aspect of the analysis later. For now, let us be content with observing in some detail how the variational approach yields the pulsational periods of stars.

2. Radial Pulsations for Self-Gravitating Systems: Stars

In this section we shall use the virial theorem, to obtain an expression for the frequency of radial pulsations in a gas sphere. The approach will be to apply a small variation to the virial theorem and by making use of several conservation laws, obtain expressions for the variation of the moment of inertia, kinetic energy, and potential energy as a function of time.

Remember from the earlier section that Lagrange's identity is

$$\tfrac{1}{2}\frac{d^2 I}{dt^2} = 2T + \Omega \,. \qquad 3.2.1$$

In this form, no time averaging has been carried out and the equation must apply to a dynamic system at any point in time. Now, consider a star with radius R. Let r be the distance from the center of symmetry to any point in the configuration and or be the displacement of a point mass from the equilibrium position r_o. Conservation of mass requires that for a spherical shell of radius

$$m(r_0 + \delta r) = m(r_0) \quad . \tag{3.2.2}$$

We wish to find the variations δI, δT, δΩ of the quantities I, T, Ω, from the equilibrium values I_0, T_0, and Ω_0. The variational form of the virial theorem then becomes

$$\tfrac{1}{2}\frac{d^2(\delta I)}{dt^2} = 2\delta T + \delta\Omega . \tag{3.2.3}$$

Since I was defined as the moment of inertia about the center of the coordinate system, we have by definition that,

$$I = \int_0^M r^2 dm(r) . \tag{3.2.4}$$

Thus, we have

$$\delta I = \int_0^M 2r\delta r dm(r) + \int_0^M r^2 \delta[dm(r)] . \tag{3.2.5}$$

Since, by the conservation of mass [equation (3.2.2)], δm(r) = 0 for all r, d(δm(r)) = δ(dm(r)) =0 for all r and the second integral of (3.2.5) vanishes leaving

$$\delta I = \int_0^M 2r\delta r dm(r) . \tag{3.2.6}$$

Now from equation (3.2.4), we have

$$\frac{dI}{dm(r)} = r^2 \quad . \tag{3.2.7}$$

Since this must always be true, it is true at the equilibrium point r_o. Therefore,

$$dI_0 = r_0^2 dm(r) . \tag{3.2.8}$$

So, to first order accuracy in r, we may re-write equation (3.2.6) as

$$\delta I = 2\int_0^{I_0} \frac{\delta r}{r_0} dI_0 \quad . \tag{3.2.9}$$

In a similar manner we may evaluate the variation of the gravitational potential energy with respect to small variation in r [3.1] and obtain

$$\delta\Omega = 2\int_0^{\Omega_0} \frac{\delta r}{r_0} d\Omega_0 . \tag{3.2.10}$$

All that now remains to be determined in equation (3.2.3) is the variation of the total kinetic energy T. To first order only the variation of the thermal kinetic energy will contribute to equation (3.2.3).[3.2]

$$2\delta T \cong 3\int_0^M \frac{P_0}{\rho_0}(\gamma-1)\frac{\delta\rho_0}{\rho_0}dm(r) \quad . \qquad 3.2.11$$

In order to facilitate obtaining an expression for $\delta\rho/\rho_0$ we shall now specify a time dependence for the pulsation about r_o. For simplicity, let us assume the motion is simply periodic. Thus, defining a quantity ξ as

$$\xi = \frac{\delta r}{r_0} = \xi_0 e^{i\sigma t} , \qquad 3.2.12$$

where $2\pi/\sigma$ is the period of oscillation, we may re-write the variations of I and Ω as follows:

$$\left.\begin{array}{l} \delta I = 2e^{i\sigma t}\int_0^{I_0}\xi_0 dI_0 \\ \delta\Omega = -e^{i\sigma t}\int_0^{\Omega_0}\xi_0 d\Omega_0 \end{array}\right\} . \qquad 3.2.13$$

Conservation of mass requires that[3.3]

$$\frac{\delta\rho}{\rho_0} = -\left(3\xi_0 + r_0\frac{d\xi_0}{dr_0}\right)e^{i\sigma t} . \qquad 3.2.14$$

Substitution of this back into the expression for the variation of the kinetic energy yields

$$2\delta T = -3\int_0^M \frac{P_0}{\rho_0}(\gamma-1)\left(3\xi_0 + r_0\frac{d\xi_0}{dr_0}\right)e^{i\sigma t}dm(r_0) . \qquad 3.2.15$$

Equation (3.2.15) may be simplified to yield[3.4]

$$2\delta T = -3\,e^{i\sigma t}\int_0^M \frac{P_0\xi_0}{\rho_0}\frac{d}{dr_0}dm(r_0) + 3\,e^{i\sigma t}\int_0^{\Omega_0}\xi_0(\gamma-1)d\Omega_0 \quad . \qquad 3.2.16$$

We now have all the material necessary to evaluate the variational form of the virial theorem to first order accuracy. Substituting equations (3.2.13) and (3.2.16) into equation (3.2.3), we obtain

$$-\sigma^2 e^{i\sigma t}\int_0^{I_0}\xi_0 dI_0 = 3e^{i\sigma t}\int_0^M \frac{P_0\xi_0 r_0}{\rho_0}\frac{d\gamma}{dr_0}dm(r) + 3e^{i\sigma t}\int_0^{\Omega_0}\xi_0(\gamma-1)d\Omega_0 - e^{i\sigma t}\int_0^{\Omega_0}\xi d\Omega_0 . \qquad 3.1.17$$

Solving for σ^2, which is related to the pulsation period, we have

$$\sigma^2 = \frac{-\int_0^{\Omega_0}(3\gamma-4)\xi_0 d\Omega_0 + 3\int_0^M \frac{P_0\xi_0 r_0}{\rho_0}\frac{d\gamma}{dr_0}dm(r)}{\int_0^{I_0}\xi_0 dI_0} \quad . \qquad 3.2.18$$

For a model of known equilibrium structure, the integrals in equation (3.2.18) may be evaluated and the frequency for which it is stable to radial pulsations may be computed. However, for purposes of examining the behavior of a pulsating star we may assume the star is sufficiently homogeneous so that γ is constant. Also, let us assume the pulsation increases radially outward in a linear manner. Under these admittedly ad hoc assumptions, equation (3.2.18) reduces to the extremely simple form

$$\sigma^2 = -\frac{(3\gamma - 4)\Omega_0}{I_0} \quad . \qquad 3.2.19$$

In order to obtain a feeling for the formula we have developed, we shall attempt to estimate some approximation pulsation frequencies. For a sphere of uniform density

$$\Omega_0 = \frac{3}{5}\frac{GM^2}{R_0} \quad . \qquad 3.2.20$$

The moment of inertia for a sphere about an axis is equal to 3/2 the moment of inertia about its center and is given by

$$I_Z = \frac{2}{5}MR_0^2 = \frac{3}{2}I_0 \quad . \qquad 3.2.21$$

Therefore,

$$I_0 = \frac{4MR_0^2}{15} \quad . \qquad 3.2.22$$

Prior theory concerning stellar structure implies that $\gamma > 4/3$. If we take $\gamma = 5/3$, appropriate for a fully convective star, we obtain

$$\sigma^2 = \frac{9}{4}\frac{GM}{R_0^3} , \qquad 3.2.23$$

or

$$\sigma^2 = 3\pi G\bar{\rho} .$$

Remembering that the period T is just $2\pi/\sigma$ we have

$$T = \left(\sqrt{\frac{4\pi}{3G}}\right)\rho^{-\frac{1}{2}} \quad . \qquad 3.2.24$$

Thus, we see that the theory does produce a period which is inversely proportional to the square root of the mean density. This law has been found to be experimentally correct in the case of the Classical Cepheids. It should be noted that this property will be preserved even for the integral form equation (3.2.18), only the constant of proportionality will change. If we evaluate the constant of proportionality from equation (3.2.24), we have

$$T \cong 7.92 \times 10^3 \bar{\rho}^{-\frac{1}{2}} \text{ sec.} \qquad 3.2.25$$

where $\bar{\rho}$ is given in (gm/cc).

Taking an observed value for the mean density of a Cepheid variable to be between 10^{-3} and 10^{-6} cm/cc (eg. Ledoux and Walraven)[4], we arrive at the following estimate for the periods of these stars.

$$0.3 \text{ days} < T < 90 \text{ days} \qquad 3.2.26$$

It is freely admitted that this estimate is arrived at in the crudest way, however, and it is comforting that the result nicely brackets the observed periods for Cepheid variables. It should also be noted that for most stars the expression arrived at in equation (3.2.23) for σ^2 is a lower limit. As the mass becomes more centrally concentrated the magnitude of the gravitational energy will increase while the moment of inertia will decrease. Even for reasonable density distributions the value arrived at in equation (3.2.23) will not differ by more than an order of magnitude. This would imply that a value for the period calculated in this manner should be correct within a factor of 2 or 3. Thus, without solving the force equations, an estimate for a very important parameter in describing the pulsation of a gas sphere may be obtained which is the period for which that sphere is stable to radial pulsation.

3 The Influence of Magnetic and Rotational Energy upon a Pulsating System

We shall now consider what the effect of introducing magnetic and rotational energies into a pulsating system will be upon the frequency of pulsation of that system. It is worth noting that solution of such a problem in terms of the force equations would be difficult indeed as it would require detailed knowledge of the geometry of the magnetic field throughout the star. However, since our approach expresses the pulsation frequency in terms of volume integrals, only knowledge of the total magnetic and rotational energies will be required.

In order to simplify the mathematical development we shall make some of the assumptions which were made during previous sections. These assumptions are listed below:

1. A first order theory will be adequate. That is, all deviations from equilibrium shall be small.
2. Radiation pressure will be considered to be negligible (i.e., $\Gamma_1 = \gamma$).
3. γ will be constant throughout the system.

We have already seen that it is possible to write Lagrange's identity so as to include the effects of rotational and magnetic energy. One of the points of that derivation that required some care was the inclusion of surface terms arising from the fact that stellar magnetic fields usually extend well beyond the normal surface of the star. However, for the moment let us neglect these terms since they usually will be small and as such will not affect the general character of the solution. Thus, as in Chapter II, we may write the scalar form of Lagrange's identity as follows:

$$\tfrac{1}{2}\frac{d^2 I}{dt^2} = 2T + \Omega + \mathcal{M}, \qquad 3.3.1$$

where T is the total kinetic energy including rotation and $\mathcal{M}$ is the total magnetic energy. Now let us break up the total kinetic energy of the system into the sum of three energies $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ where $\mathcal{T}_1$ is the kinetic energy of the pulsating system due to the pulsating motion, $\mathcal{T}_2$ is the kinetic energy of the gas due to thermal energy, and $\mathcal{T}_3$ is the kinetic energy of rotation. The contribution to the kinetic energy (of particle motion) due to an element of mass is

$$d\mathcal{T}_2 = \frac{3}{2}kT\mathrm{dn} = \frac{3}{2}RT\mathrm{dm} = \frac{3}{2}(c_p - c_v)T\mathrm{dm}\,, \qquad 3.3.2$$

where T and R are the gas temperature and constant respectively. But the internal energy $d\mathcal{U}$ of the element of mass is

$$d\mathcal{U} = c_v T\mathrm{dm}\,, \qquad 3.3.3$$

Combining equation (3.3.2) and equation (3.3.3), with the definition of γ, we have

$$d\mathcal{T}_2 = \frac{3}{2}(\gamma-1)d\mathcal{U}\,. \qquad 3.3.4$$

Integrating this over the entire system we obtain

$$2\mathcal{T}_2 = 3(\gamma-1)\mathcal{U}\,. \qquad 3.3.5$$

Thus, we may write the virial theorem for the system as

$$\tfrac{1}{2}\frac{d^2I}{dt^2} = 2\mathcal{T}_1 + 2\mathcal{T}_3 + 3(\gamma-1)\mathcal{U} + \Omega + \mathcal{M}\quad . \qquad 3.3.6$$

The variational form becomes

$$\tfrac{1}{2}\frac{d^2}{dt^2}(\delta I) = 2\delta\mathcal{T}_3 + 3(\gamma-1)\delta\mathcal{U} + \delta\Omega + \delta\mathcal{M}\,. \qquad 3.3.7$$

There is no term containing a variation in γ as it is assumed to be constant throughout the system. In section 2, it was shown that variation of the pulsational kinetic energy was of second order and could therefore be neglected. Thus

$$2\delta\mathcal{T}_1 = 0\,. \qquad 3.3.8$$

Since we have already computed the variation of the kinetic energy of the gas, we may most easily find the variation of the total internal energy ($\delta\mathcal{U}$) in terms of this quantity. As a result of the assumption of constant γ, we may take the variation of equation (3.3.5), and obtain

$$2\delta\mathcal{T}_2 = +3(\gamma-1)\delta\mathcal{U}\,. \qquad 3.3.9$$

Now, if we further assume a periodic form for the pulsation and linearly increasing amplitude (ξ_0=const.), equation (3.2.15), gives the following expression for the variation of the kinetic energy of the gas:

$$2\delta\mathcal{T}_2 = -3\int_0^M \frac{3P_0\xi_0}{\rho_0}(\gamma-1)e^{i\sigma t}\mathrm{dm}(r_0)\,. \qquad 3.3.10$$

or $$2\delta\mathcal{T}_2 = -3(\gamma-1)\xi_0 e^{i\sigma t}\int_V 3P_0 dV .$$

Combining equation (3.3.10) and equation (3.3.9), we obtain

$$\delta\mathcal{U} = -\xi_0 e^{i\sigma t}\int_V 3P_0 dV . \qquad 3.3.11$$

In order to express this variation in terms of the other energies present in the system, we shall assume that the system is in quasi-steady state. With this assumption, the relevant quantities are averaged over one pulsation period so that the $< d^2I/dt^2 >= 0$. We shall assume that the remaining values are the equilibrium values of the configuration. The virial theorem as expressed in equation (3.3.6) then becomes

$$2\mathcal{T}_1(0) + 2\mathcal{T}_2(0) + 2\mathcal{T}_3(0) + \Omega_0 + \mathcal{M} = 0 . \qquad 3.3.12$$

Now, from the elementary kinetic theory of gases we have

$$\mathcal{T}_2 = \tfrac{3}{2}\int_V P_0 dV . \qquad 3.3.13$$

Also, since the system is neither expanding or contracting,

$$\mathcal{T}_1(0) =< \mathcal{T}_1 >= 0 . \qquad 3.3.14$$

Making use of these two results and replacing the average values of the quantities in equation (3.3.12) with the equilibrium values we find

$$-3\int_V P_0 dV = 2\mathcal{T}_3(0) + \Omega_0 + \mathcal{M}_0 . \qquad 3.3.15$$

Identifying the left side of equation (3.3.15) with the right side of equation (3.3.11), we finally obtain

$$\delta\mathcal{U} = \xi_0 e^{i\sigma t}\left(2\mathcal{T}_3(0) + \Omega_0 + \mathcal{M}_0\right) . \qquad 3.3.16$$

We have already determined expressions for the variations of the gravitational energy and moment of inertia in the previous section [equation (3.2.13)]. Under the assumption used above, of a constant perturbation ξ_o, these equations become

$$\left.\begin{aligned} \delta I &= 2e^{i\sigma t}\xi_0 I_0 \\ \delta\Omega &= -e^{i\sigma t}\xi_0\Omega_0 \end{aligned}\right\} . \qquad 3.3.17$$

Thus, we need only obtain expressions for the variation of the magnetic and rotational energies in order to evaluate equations (3.3.7).

Consider first the rotational energy. Now an element of mass rotating about an axis with a velocity ω will possess an elemental angular momentum

$$d\mathcal{L} = \omega(x^2 + y^2)dm(r) = \omega r^2 \sin^2\theta\, dm(r) . \qquad 3.3.18$$

Here the x-y plane is the plane perpendicular to axis of rotation and θ is the polar angle measured from the axis of rotation. Such an elemental mass possessing such an angular momentum will have a rotational kinetic energy given by

$$d\mathcal{T}_3 = \tfrac{1}{2}\omega d\mathcal{L} . \qquad 3.3.19$$

Thus, the total rotational kinetic energy is just

$$\mathcal{T}_3 = \tfrac{1}{2}\int_0^{\mathcal{L}} \omega d\mathcal{L} . \qquad 3.3.20$$

We may use this expression to obtain the variation $\delta\mathcal{T}_3$. The first term on the right of equation (3.3.7) becomes

$$2\delta\mathcal{T}_3 = \delta\int_0^{\mathcal{L}} \omega d\mathcal{L} = \int_0^{\mathcal{L}} \delta\omega d\mathcal{L} + \int_0^{\mathcal{L}} \omega d(\delta\mathcal{L}) . \qquad 3.3.21$$

However, the conservation of angular momentum requires that $\mathcal{L}$ remain constant during the pulsation. Thus, the variation of $\mathcal{L}$ is zero and the last integral on the right of equation (3.3.21) vanishes, so that

$$2\delta\mathcal{T}_3 = \int_0^{\mathcal{L}} \delta\omega d\mathcal{L} + \int_0^{\mathcal{L}} \frac{\delta\omega}{\omega_0}\,\omega_0 d\mathcal{L} . \qquad 3.3.22$$

where ω_o is the rotational velocity of the equilibrium configuration.

Now, again making use of the conservation of angular momentum we see that

$$\omega r^2 \sin\theta = \text{const}. \qquad 3.3.23$$

Since we are only considering radial pulsation so that δθ is zero, the variation equation (3.3.23) yields

$$\delta\omega r^2 \sin\theta + 2\omega r\delta r \sin\theta = 0 \quad . \qquad 3.3.24$$

This is equivalent to "conserving angular momentum in shells."

If we evaluate equation (3.3.24) at the equilibrium position, we obtain an expression of first order accuracy,

$$\frac{\delta\omega_0}{\omega_0} = \frac{2\delta r}{r_0} = -2\xi_0 e^{i\sigma t} . \qquad 3.3.25$$

Substitution of this expression into equation (3.3.22) yields

$$2\delta\mathcal{T}_3 = -\int_0^{\mathcal{L}} \left(2e^{i\sigma t}\xi_0\omega_0\right) d\mathcal{L}_0 . \qquad 3.3.26$$

If, for simplicity, we further assume that the rotational velocity is a constant throughout the configuration. We obtain a very simple form of the variation of the rotational energy.

$$2\delta\mathcal{T}_3 = -\left(2e^{i\sigma t}\xi_0\omega_0\right)\mathcal{L}_0 , \qquad 3.3.27$$

where $\mathcal{L}_o$ is the total angular momentum for the system. Thus, only the variation of the magnetic energy remains to be determined.

In order to determine the variation of the total magnetic energy it is necessary to establish a coordinate system appropriate to the geometry of the field and to the geometry of the configuration. Although the configuration is spherically symmetric, the geometry of the magnetic field present is not known. Thus, we shall consider the variations in Cartesian coordinates and later reduce our result to a form which is compatible with our previous results. Now the total magnetic energy of the configuration is defined (in c.g.s. units).

$$\mathcal{M} = \int_0^M \frac{|\mathbf{H}|^2}{8\pi\rho} dm(r) . \qquad 3.3.28$$

Thus, denoting the Cartesian coordinates as x_1, x_2, and x_3, the variational form of the magnetic energy is

$$\delta\mathcal{M} = \frac{1}{4\pi}\int_0^M \frac{\mathbf{H}\bullet\delta\mathbf{H}}{\rho} dm(r) - \frac{1}{8\pi}\int_0^M |\mathbf{H}|^2 \frac{\delta\rho}{\rho^2} dm(r) , \qquad 3.3.29$$

or in Cartesian coordinates

$$\delta\mathcal{M} = \frac{1}{4\pi}\iiint \sum_i H_i \delta H_i dx_1 dx_2 dx_3 - \frac{1}{8\pi}\iiint |\mathbf{H}|^2 \frac{\delta\rho}{\rho} dx_1 dx_2 dx_3 \ . \qquad 3.3.30$$

Although we have already obtained an expression for $\delta\rho/\rho$ in section 3, due to the introduction of Cartesian coordinates it is convenient to express this variation in terms of the variation of the coordinates η_i [3.5], namely

$$\frac{\delta\rho}{\rho} = -\sum_{i=1}^{3} \frac{\partial\eta_i}{\partial x_i} \ , \qquad 3.3.31$$

Before we can evaluate the expression for the variation of the magnetic energy, we must first determine the variation of the magnetic field δH_i. A rather lengthy argument[3.6] shows we can express this in terms of the coordinate variations, so

$$\delta H_i = \sum_j \left(H_j \frac{\partial\eta_i}{\partial x_j} - H_i \frac{\partial\eta_j}{\partial x_i} \right) \ . \qquad 3.3.32$$

If we substitute (3.3.32) and (3.3.31) into (3.3.29), we have

$$\begin{aligned}\delta\mathcal{M} = &\frac{1}{4\pi}\iiint \sum_i \sum_j H_i H_j \frac{\partial\eta_i}{\partial x_j} dx_1 dx_2 dx_3 - \frac{1}{4\pi}\iiint \sum_i \sum_j H_i^2 \frac{\partial\eta_j}{\partial x_j} dx_1 dx_2 dx_3 \\ &+ \frac{1}{8\pi}\iiint H^2 \sum_j \frac{\partial\eta_j}{\partial x_j} dx_1 dx_2 dx_3\end{aligned} \ . \qquad 3.3.33$$

The second and third terms combine to yield

$$\delta\mathcal{M} = \frac{1}{4\pi}\iiint \sum_i \sum_j H_i H_j \frac{\partial\eta_i}{\partial x_j} dx_1 dx_2 dx_3 + \frac{1}{8\pi}\iiint H^2 \sum_j \frac{\partial\eta_j}{\partial x_j} dx_1 dx_2 dx_3 . \qquad 3.3.34$$

Now, if we assume that η_i is only a function of x_i then the sum on j in the first term collapses and the remaining terms result in

$$\delta\mathcal{M} = \frac{1}{8\pi}\iiint\left(\sum_i 2H_i^2\frac{\partial\eta_j}{\partial x_j} - |\mathbf{H}|^2\frac{\partial\eta_i}{\partial x_i}\right)dx_1dx_2dx_3 . \qquad 3.3.35$$

At this point, it is appropriate to re-introduce the assumption concerning the nature of the variation η_i. It was earlier assumed that ξ_o was constant. The equivalent assumption for the η_i's is that

$$\eta_i = \text{const } x_i \quad . \qquad 3.3.36$$

Substitution of this explicit variation into equation (3.3.35) yields

$$\delta\mathcal{M} = -\frac{\text{const.}}{8\pi}\iiint|\mathbf{H}|^2 dx_1dx_2dx_3 . \qquad 3.3.37$$

Now, since we wish to consider the same type of pulsation from the η_i's, as we have assumed in the earlier section, we require that

$$\eta = \delta\mathbf{r} . \qquad 3.3.38$$

or

$$\frac{\eta}{r} = \xi = \xi_0 e^{i\sigma t}\hat{\mathbf{r}} . \qquad 3.3.39$$

Making use of our definition for η_i's, equation (3.3.36), we have

$$\text{const.}\left(\frac{x_i + x_j + x_k}{r}\right) = \xi = \xi_0 e^{i\sigma t} . \qquad 3.3.40$$

This relation can only be true if the pulsation in the three coordinates (η_i) are in phase and of equal amplitude, and if

$$\text{const.} = \xi_0 e^{i\sigma t} . \qquad 3.3.41$$

Now, using the definition for the mass in a given volume in Cartesian coordinates and the value for the constant in equation (3.3.37), we may re-write the variation of the magnetic energy as follows:

$$\delta\mathcal{M} = -\xi_0 e^{i\sigma t}\left(\frac{1}{8\pi}\int_0^M \frac{H^2}{\rho}dm(r)\right) . \qquad 3.3.42$$

Making use of equation (3.3.28) we may rewrite the variation in terms of the total magnetic energy of the equilibrium configuration

$$\delta\mathcal{M} = -\xi_0 e^{i\sigma t}\mathcal{M}_0 \quad . \qquad 3.3.43$$

Thus, we have obtained an expression for the last variation required to evaluate the variational form of the virial theorem (equations 3.3.7). We may, therefore, substitute equations (3.3.16), (3.3.27), and (3.3.43), into equations (3.3.7), and obtain

$$\tfrac{1}{2}\frac{d^2}{dt^2}\left(2e^{i\sigma t}\xi_0 I_0\right) = -2e^{i\sigma t}\xi_0\omega_0\mathscr{L}_0 + 3(\gamma-1)\left(\xi_0 e^{i\sigma t}\right)\left(\omega_0\mathscr{L}_0 + \Omega_0 + \mathcal{M}_0\right) - \xi_0 e^{i\sigma t}\Omega_0 - \xi_0 e^{i\sigma t}\mathcal{M}_0 \, . \qquad 3.3.44$$

Simplifying equation (3.3.44), we find that

$$\sigma^2 = \frac{-(3\gamma-4)(\Omega_0+\mathcal{M}_0) + (5-3\gamma)\omega_0\mathscr{L}_0}{I_0} \quad . \qquad 3.3.45$$

Although we have made some strict assumptions in deriving equation (3.3.45), one should not feel that they are all of paramount importance. The assumption of constancy of ξ_o and γ were only made so that the resultant integrals could be integrated in terms of the original parameters. If necessary, these assumptions may be omitted and an integral expression similar to equation (3.2.18) may be derived. However, the work required to obtain this expression is non-trivial and in order for it to be useful one must have a detailed model in mind. One must also know the detailed geometry of the magnetic fields and of the star in order to evaluate the integrals that result. Also, to study the behavior of σ^2 , a great deal of numerical work will be required. For purposes of studying the effects of changes in γ, Ω_o, M_o, ω_0, and $\mathscr{L}_0$, equation (3.3.43) will be quite adequate and is much easier to handle. Equation (3.3.45) contains many aspects which one may check for 'reasonableness'. If we let η_o and ω_0 be zero, then equation (3.3.45) becomes identical to the previously derived equation (3.2.19). Letting only ω_0 be zero we obtain an expression identical to one arrived at by Chandrasekhar and Limber[5] (1954). If ω_0 is non-zero, while $\mathcal{M}_o$ is zero, the expression is that of Ledoux[1] (1945). It should not be surprising to find the magnetic energy entering in an additive manner to the gravitational energy. Both are potential energies and since the basic equations are scalar in nature, we should expect the final result to merely 'modify' the gravitational energy. However, the rotational energy is kinetic in nature and hence would not enter into the final result in the same manner as the magnetic energy. Let us briefly investigate the effect upon σ^2, and hence on the pulsational period of the presence of magnetic and rotational energy. Since,

$$T = 2\pi/\sigma \, , \qquad 3.3.46$$

an increase in σ indicates a decrease in the period and vice-versa. Now, since $\gamma > 4/3$ and the gravitational potential energy is defined as being negative, the first term in the numerator of equation (3.3.45) will be positive only if

$$|\Omega_0| > \mathcal{M}_0 \, . \qquad 3.3.47$$

Thus, the introduction of magnetic fields only serves to reduce σ^2 and thereby lengthen the period of pulsation. However, the addition of rotational energy (ω_0 $\mathscr{L}_0$) will tend to increase σ^2 as long as $\gamma > 5/3$. When $\gamma = 5/3$, the introduction of rotation has no effect on the period of pulsation. If $\gamma < 5/3$, the influence of rotation is similar to that of magnetic energy.

4. Variational Form of the Surface Terms

In deriving the virial theorem, we noted earlier that the use of the divergence theorem yields some surface integrals which are generally ignored. Formally they may be ignored by taking the bounding surface of the configuration to be at infinity. However, in reality, this generally proves to be inconvenient for stars as they usually have a reasonably well-defined surface or boundary. For stars possessing general magnetic fields which extend beyond the surface, these surface contributions should be included. They are usually wished away by assuming they are small compared to the total magnetic energy arising from the volume integration. Although this may be true for simple fields in stars, it is unlikely to be true for other gaseous configurations such as flares and in any event a numerical estimate of their importance is far more re-assuring than an intuitive feeling. For this reason, let us consider the way in which these surface terms affect the variational formalism of the previous section. To facilitate the calculations, we will assume the star is nearly spherical and the pulsations are radial. If the magnetic field is strong this will clearly not be the case and the full tensor virial theorem must be used. However, the simplicity generated by the use of the scalar virial theorem justifies the approach for purposes of illustration. Let us begin by sketching the origin of the virial theorem as rigorously presented by Chandrasekhar[6]. The equations of motion for a gas with zero resistivity are

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla\rho + \rho\nabla\Phi + \frac{1}{4\pi}\left(\nabla\times\mathbf{H}\right)\times\mathbf{H}\,. \qquad 3.4.1$$

Employing the identity $(\nabla\times\mathbf{H})\times\mathbf{H} = (\mathbf{H}\bullet\nabla)\mathbf{H} - \nabla(\mathbf{H}\bullet\mathbf{H})/2$ and taking the scalar product of equation (3.4.1) with the position vector **r** then integrating over all space enclosed by the bounding surface, we get

$$\int_V \rho\mathbf{r}\bullet\frac{d\mathbf{u}}{dt}dV = -\int_V \mathbf{r}\bullet\nabla P dV + \int_V \rho\mathbf{r}\bullet\nabla\Phi dV + \frac{1}{4\pi}\int_V \mathbf{r}\bullet(\mathbf{H}\bullet\nabla)\mathbf{H}dV - \frac{1}{8\pi}\int_V \mathbf{r}\bullet\nabla(H^2)dV\,. \qquad 3.4.2$$

As in section 3, this becomes

$$\tfrac{1}{2}\frac{d^2I}{dt^2} - 2T = 3(\Gamma-1)\mathcal{U} + \Omega + \mathcal{M} - \int_S P_0\mathbf{r}\bullet d\mathbf{S} - \frac{1}{8\pi}\int_S H_0^2\mathbf{r}\bullet d\mathbf{S} + \frac{1}{4\pi}\int_S (\mathbf{r}_0\bullet\mathbf{H}_0)(\mathbf{H}_0\bullet d\mathbf{S})\,, \qquad 3.4.3$$

where P_o and $\mathbf{H}_o$ are the gas pressure and magnetic field present at the surface $\mathbf{r}_o$. It is the behavior of the three integrals in equation (3.4.3) that will interest us as hopefully the remaining terms are by now familiar. Consider first the effect of a pulsation on the surface term arising from the pressure by taking the variations of the surface pressure integral.

$$\delta\int_S P_0\mathbf{r}_0\bullet d\mathbf{S} = \int_S \delta P_0\mathbf{r}_0\bullet d\mathbf{S} + \int_S P_0\delta\mathbf{r}_0\bullet d\mathbf{S} + \int_S P_0\mathbf{r}_0\bullet d(\delta\mathbf{S})\,. \qquad 3.4.4$$

For radial variations only

$$d(\delta\mathbf{S}) = 2r_0\delta r_0\sin\theta\, d\theta d\phi = 2\xi r_0^2\sin\theta\, d\theta d\phi = 2\xi dS\,, \qquad 3.4.5$$

where, as in section 2, $\xi = \delta r/r$. In Chapter III, section 2 [equations (N3.2.13) and (3.2.14)], we already have shown that for adiabatic pulsations

$$\frac{\delta P}{P} = \gamma\frac{\delta\rho}{\rho} = -\gamma\left(3\xi + r\frac{d\xi}{dr}\right) \quad . \tag{3.4.6}$$

Combining equation (3.4.5) and equation (3.4.6) with equation (3.4.4), we get

$$\delta\int_S P_0\mathbf{r}_0 \bullet d\mathbf{S} = 3(\gamma-1)\int_S \xi P_0\mathbf{r}_0 \bullet d\mathbf{S} - \gamma\int_S\left(r_0\left.\frac{d\xi}{dr}\right|_{r_0}\right)\mathbf{r}_0 \bullet d\mathbf{S}\,\mathbf{.} \tag{3.4.7}$$

Earlier we assumed that ξ was constant throughout the star and hence its derivative vanished. Here, we only require the derivative to vanish at the surface in order to simplify equation (3.4.7) to get

$$\delta\int_S P_0\mathbf{r}_0 \bullet d\mathbf{S} = 3(\gamma-1)\xi\int_S P_0 r_0 dS\,\mathbf{.} \tag{3.4.8}$$

Now consider the variation of the two magnetic integrals in equation (3.4.3).

$$\begin{aligned}\delta\left[\frac{1}{8\pi}\int_S 2(\mathbf{r}_0 \bullet \mathbf{H}_0)(\mathbf{H}_0 \bullet d\mathbf{S}) - \frac{1}{8\pi}\int_S H_0^2\mathbf{r}_0 \bullet d\mathbf{S}\right] &= \frac{1}{8\pi}\int_S 2(\delta\mathbf{r}_0 \bullet \mathbf{H}_0)(\mathbf{H}_0 \bullet d\mathbf{S}) + \frac{1}{8\pi}\int_S 2(\mathbf{r}_0 \bullet \delta\mathbf{H}_0)(\mathbf{H}_0 \bullet d\mathbf{S}) \\ &+ \frac{1}{8\pi}\int_S 2(\mathbf{r}_0 \bullet \mathbf{H}_0)(\delta\mathbf{H}_0 \bullet d\mathbf{S}) + \frac{1}{8\pi}\int_S 2(\mathbf{r}_0 \bullet \mathbf{H}_0)(\mathbf{H}_0 \bullet d\delta\mathbf{S}) + \frac{1}{8\pi}\int_S 2(\mathbf{H}_0 \bullet \delta\mathbf{H}_0)(\mathbf{r}_0 \bullet d\mathbf{S}) \\ &- \frac{1}{8\pi}\int_S H_0^2\delta\mathbf{r}_0 \bullet d\mathbf{S} - \frac{1}{8\pi}\int_S H_0^2\mathbf{r}_0 \bullet d\delta\mathbf{S}\end{aligned} \tag{3.4.9}$$

This truly horrendous expression does indeed simplify[3.7] by using

$$\delta\mathbf{H} = -2\xi\mathbf{H}_0\,\mathbf{.} \tag{3.4.10}$$

Using this result, equation (3.4.5) and the definition of ξ, equation (3.4.9) becomes:

$$\delta Q_m = -\frac{\xi}{8\pi}\left[\int_S 2(\mathbf{r}_0 \bullet \mathbf{H}_0)(\mathbf{H}_0 \bullet d\mathbf{S}\right] - \int_S H_0^2\mathbf{r}_0 \bullet d\mathbf{S} \quad \mathbf{.} \tag{3.4.11}$$

where Q_m stands for the original magnetic surface term that appears in equation (3.4.3). Thus, the variation of the surface terms can be represented as

$$\left.\begin{aligned}\delta Q_m &= -\xi Q_m \\ \delta Q_P &= 3(\gamma-1)\xi Q_P\end{aligned}\right\}\mathbf{.} \tag{3.4.12}$$

If we assume a linear or homologous pulsation, as was done in the previous two sections, then the expression for the pulsational frequency [equation (3.3.45)], becomes

$$\sigma^2 = -\frac{(3\gamma-4)(\Omega_0 + \mathcal{M}_0) - (5-3\gamma)\omega_0\mathcal{L}_0 + (3\gamma-1)Q_P - Q_m}{I_0} \quad \mathbf{.} \tag{3.4.13}$$

Since $\gamma > 4/3$, the contribution of the surface pressure term is such as to increase σ^2 and thereby improve the stability of the system. Basically, this results because an unstable system will have to do work against the surface pressures either in expanding or contracting the surface. This energy is thus not available to feed the instability. The situation is not as obvious for the magnetic contribution Q_m , since Q_m is the difference between two positive quantities. Thus, the result depends entirely on the geometry of the field. The effect of the field geometry can be made somewhat clearer by considering a spherical star so that the radius vector is parallel to the surface normal. Under these conditions Q_m becomes

$$Q_m = \frac{1}{8\pi}\int_S \left[2(\hat{\mathbf{H}}_0 \bullet \hat{\mathbf{r}})^2 - 1\right] H_0^2 r dS = \frac{1}{8\pi}\int_S \cos\beta\, H_0^2 r dS \quad , \qquad 3.4.14$$

where β is the local angle between the field and the radius vector. Thus, the average of $\cos 2\beta$ weighted by H_0^2 over the surface will determine the sign of Q_m. In any event, it is clear that

$$|Q_m| < \frac{1}{8\pi}\int_S H_0^2 r_0 dS = \tfrac{1}{2} R_0^3 \overline{H_0^2} \; . \qquad 3.4.15$$

It is worth noting that in the case where the magnetic field increases slowly with depth, this term can be of the same order of magnitude as the internal magnetic field energy and must be included. Furthermore, whether or not the local contribution to Q_m is positive or negative depends on whether or not the local value of β is greater or less than $\pi/4$. Since a positive value of Q_m increases the value of σ^2, fields exhibiting a local angle to the radius vector greater than $\pi/4$ tend to stabilize the object, whereas more radial fields enhance the instability. This simply results from the fact that a radial motion will tend to compress fields more nearly tangential to the motion than 45^o, thereby removing energy from the motion. Conversely, more nearly radial fields will tend to feed the perturbation leading to a decrease in stability.

One thing becomes immediately clear from this discussion. If Q_m is an important term in equation (3.4.13), radial pulsation will not occur. Since a magnetic field cannot exhibit spherical symmetry, the departures from symmetry will yield a variable "restoring force" over the surface inferring that non-radial modes will be excited. In this case the tensor virial theorem must be used and the field geometry known. Lastly, for purposes of simplicity, we have assumed no coupling between the gas pressures and magnetic pressures. Unless the system is rather bizarre, the gas will be locally relaxed on a time scale less than the pulsation period and hence the two cannot be treated independently. This assumption was made merely for the sake of simplicity and doesn't affect the illustrative aspects of the effects. However, unless Q_m and Q_P are comparable the coupling between the two will be weak and we may expect equation (3.4.13) to give good quantitative results.

5. The Virial Theorem and Stability

In the last section, I alluded to the effects that the surface terms have on the stability of the system being considered. This concept deserves some amplification as it represents one of the most productive applications of the virial theorem. However, before embarking on a detailed development of the virial theorem for this purpose, it is appropriate to review the use of the word stability itself.

When inquiring into the meaning of the word, it is customary to consult a dictionary. This approach provides the following definition:

> *Stability: "That property of a body which causes it, when disturbed from a condition of equilibrium or steady motion, to develop forces or moments which tend to restore the body to its original condition."*

This definition is subject to several interpretations and serves to illustrate the danger of consulting an English dictionary to learn the meaning of a technical term. The word stability is usually associated with the word equilibrium. This is primarily because the concept of stability normally is first encountered during the study of statistics. However, there are many dynamical situations, which are not equilibrium situations that even the most skeptical person would call stable. One of the most obvious examples to an astronomer, are the stars themselves. Not all stars would be regarded as stable, but certainly most of the main sequence stars are. Since stars are not really equilibrium configurations, but rather steady state configurations, we see that we must extend our conceptualization of stability to include some dynamical systems. The normal definition of equilibrium requires that the sum of all forces acting on the system is zero. This concept may be broadened to dynamical systems if one requires that the generalized forces (Q_i) acting on the systems are zero. Here the concept of the generalized force may be most simply stated as

$$Q_i = \sum_j \mathbf{F}_j \bullet \frac{\partial \mathbf{r}}{\partial q_i} , \qquad 3.5.1$$

where $\mathbf{F}_j$ represents the physical forces of the system acting on the jth particle and the q_i's represent any set of linearly independent 'coordinates' adequate to describe the system. In a conservative system all the forces are derivable from a potential Φ .Thus, the generalized forces may be written as

$$Q_i = -\sum_j \nabla_j \Phi \bullet \frac{d\mathbf{r}_j}{dq_i} = \sum_j \left(\frac{\partial \Phi}{\partial r_j}\right)\hat{\mathbf{r}}_j \bullet \left(\frac{\partial \mathbf{r}_j}{\partial q_i}\right) = -\frac{\partial \Phi}{\partial q_i} \;. \qquad 3.5.2$$

Thus, saying the generalized forces must vanish is equivalent to saying the potential energy must be in extremum.

$$Q_i = -\left(\frac{\partial \Phi}{\partial q_i}\right)\Bigg|_{q_i = q_i(0)} = 0 \,. \qquad 3.5.3$$

Now, in terms of this definition of equilibrium we may proceed to a definition of stable equilibrium. If the potential extremum implied by equation (3.5.3) is a minimum, then the equilibrium is said to be stable. The conditions thus imposed on the potential are

$$Q_i = -\left(\frac{\partial^2 \Phi}{\partial q_i^2}\right)\Bigg|_{q_i = q_i(0)} > 0 \,. \qquad 3.5.4$$

In order to see that this definition of stability is consistent with our dictionary definition, consider the following argument. Suppose a system is disturbed from equilibrium by an increase in the total energy dE above the total energy at equilibrium. If Φ is a minimum, any disturbances from equilibrium will produce an increase in the potential energy. Since the conservation of energy will apply to the system after the incremental energy dE has been applied, the kinetic energy must decrease. This implies that the velocities will decrease for all particles and eventually become zero. Thus, the motion of the system will be bounded *(Note: the bound may be arbitrarily large)*. If, however, the departure from equilibrium brings about a decrease in the potential energy, then the velocities may increase without bound. We would certainly call such motion unstable motion.

However, simple and clear-cut as this definition of stability may seem, it is still inadequate to serve the needs of mathematical physicists in describing the behavior of systems of particles. Thus it is not uncommon to find modifying adjectives or compound forms of the word "stable" appearing in the literature. A few common examples are: Secular stability, global stability, quasi-stable, bi-stable, and over-stable. These terms are usually used without definition in the hope that the reader will be able to discern the correct meaning from the context. The introduction of these modifiers as often as not arises from the mode of analysis used to describe the system. It is a common practice to examine the response of the system to a continuous spectrum of perturbations. If any of these perturbations grow without bound the system is said to be unstable. This would seem in full accord with our dictionary definition and thereby wholly satisfying. Unfortunately, one is rarely able to calculate the response in general. It is usually necessary to linearize the equations describing the system in order to solve them. Analysis of this type is called linear stability theory, and is actually the basis for most stability criterion. Thus, when analyzing a system not only must one correctly carry out the stability analysis, he must also decide on the applicability of the analysis to the system.

Recently, it has been quite fashionable to use the virial theorem as the vehicle to carry out linearized normal mode analysis of systems in order to determine their state of stability. However, the determination of a system state of stability seems to have inspired Jacobi to develop the n-body representation of Lagrange's identity from which it is a short step to the virial theorem. To see now how closely tied the virial theorem is connected to stability; let us summarize some of Jacobi's arguments. In Chapter I, [i.e. equation (1.4.12)], we arrived at simple statements of Lagrange's identity for self-gravitating systems as:

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2T + \Omega \,. \qquad 3.5.5$$

One could say with some confidence that if $d^2I/dt^2 > 0$ for all t the system would have to have at least one particle whose position coordinates increased without bound. That is to say, the system would be unstable. However, since both T and Ω vary with time it would be difficult to say something a priori about d^2I/dt^2 from Lagrange's identity alone. Thus, Jacobi employed the constancy of the total energy (i.e., $E = T + \Omega = \text{const.}$), and the fact that for self-gravitating systems $\Omega > 0$, to modify equation (3.5.5) to give:

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2E - \Omega > 2E \,. \qquad 3.5.6$$

So, if $E > 0$, $d^2I/dt^2 > 0$ and the system is unstable. This is known as Jacobi's stability criterion and provides a sufficient (but not necessary) condition for a system to be called unstable.

It is the constancy of E with time that makes this a valuable criterion for stability and is the reason Jacobi used it. There is another approach to the problem of temporal variation which was not available to Jacobi. In Chapter 2 we discussed the extent to which time averages of quantities may be identified within their phase averages, through use of the Ergodic theorem. Integrating equation (3.5.5) over some time t , we get

$$\frac{1}{t_0}\left(\frac{dI_0}{dt}\right)\Bigg|_0^{t_0} = 2\overline{T} + \overline{\Omega} \quad . \qquad 3.5.7$$

If the system is to remain bounded, dI/dt must always be finite. If the system is to be always stable, then the limit of the left-hand side must tend to zero. Furthermore, the time averages on the right-hand side of (3.5.7) will tend to phase averages if the system is Ergodic. Thus

$$2\langle T\rangle + \langle Q\rangle = 0, \qquad 3.5.8$$

constitutes a stability criterion for Ergodic systems. That is, equation (3.5.8) must be satisfied in a stable Ergodic system and failure of equation (3.5.8) is sufficient for instability. It is not uncommon to find the statement in the literature that ($2T + Q > 0$) insures the instability of a system citing the virial theorem as the justification. Actually, it is Lagrange's identity that is the relevant expression and it only guarantees that at the moment the system is acceleratively expanding. What is really meant is that if ($T + Q > 0$) the system is indeed unstable as this is just a statement of Jacobi's stability criterion concerning the total energy of the system.

Now, let us turn the applications of the variational form of the virial theorem to stability, keeping in mind that the variational approach is essentially a first order or linearized analysis. The majority of Chapter III has been devoted to obtaining expressions for the frequency of a pulsating system. We obtained a value for the square of the frequency in terms of the equilibrium energies of the configuration. However, these expressions could be neither positive nor negative. In the earlier sections, we discussed only the meaning of the positive values, as negative squares of frequencies had no apparent physical meaning. Let us look again at the nature of the assumed pulsation in order to further investigate the meaning of these pulsation frequencies. In section 2 [i.e. equations (3.2.12)], we assumed that the pulsation would be periodic and of the form

$$\frac{\delta r}{r} = \xi_0 e^{i\sigma t} \quad . \qquad 3.5.9$$

where σ was the frequency of pulsation and ξ_o did not depend on time. Now, if we make the formal identification between σ and the period and take σ to be purely imaginary, we may write

$$\sigma = \pm 2\pi i / t_0 , \qquad 3.5.10$$

where t_o is a real number. Combining equation (3.5.10) and equation (3.5.9), we have

$$\frac{\delta r}{r} = \xi_0 e^{\pm 2\pi t / t_0} . \qquad 3.5.11$$

Thus, the pulsation becomes exponential in nature. If the sign of σ is negative, then the sign of the exponential in equation (3.5.11) will be positive and the pulsation will grow without bound with a rate of growth determined by t_o.

One might be tempted to choose the negative sign of equation (3.5.11) saying that the system is stable as the pulsation will die out, even though $\sigma^2 < 0$. This would be wrong. All classical equations of dynamical symmetry exhibit full-time symmetry, thus those solutions which damp out in the future were unstable in the past and vice-versa. A specific solution would be fully determined by the boundary conditions at t = 0. Further, we assumed that a full continuum of perturbations are present, resulting from small but inevitable, departures from perfection produced by statistical fluctuations. Thus, if there exists even one mode with $\sigma^2 < 0$, the instability associated with that mode will grow without bound. Therefore, this becomes a sufficient condition for the system to be unstable in the strictest sense of the word

$$\sigma^2 < 0. \qquad 3.5.12$$

It is also worth noting that this criterion applies to the entire system, and thus is a "global" stability condition. However, it can be made into a local condition by taking an infinitesimal volume and including the surface terms discussed in section 4.

Now, let us see what implications this analysis has for the stability of stars. In section 2, we established an expression for the pulsation frequency of a gravitating gas sphere [equation (3.2.19)]. Now, applying the instability criterion equation (3.5.12), we see that the sphere will become unstable, when

$$-\frac{(3\gamma - 4)\Omega_0}{I_0} < 0 \ . \qquad 3.5.13$$

Since the moment of inertia (I_0) is intrinsically positive, while the gravitational potential energy is intrinsically negative, equation (3.5.13) becomes

$$\left. \begin{array}{l} (3\gamma - 4) < 0 \\ \gamma < 4/3 \end{array} \right\} . \qquad 3.5.14$$

Thus, a star will become unstable when γ is less than 4/3. This is the familiar instability criterion demonstrated by Chandrasekhar in *Stellar Structure*.[7] He further demonstrates that a gas with γ equal to 4/3 corresponds to a gas where the total pressure is entirely due to radiation. If we consider the other stability criterion [equation (3.5.14)], we can see that a necessary condition for stability of a homogeneous non-rotating gas sphere is

$$\gamma > 4/3 . \tag{3.5.15}$$

Thus, having extracted as much information as possible from the pulsation expression developed in section 2, let us turn to the more general formulae resulting from our analysis in section 3. Remember the final expression for the pulsational frequency was

$$\sigma^2 = \frac{-(3\gamma-4)(\Omega_0+\mathcal{M}_0)+(5-3\gamma)\omega_0\mathcal{L}_0}{I_0} . \tag{3.5.16}$$

Consider first a gas sphere which is not rotating, but which has a magnetic field. Equation (3.5.16) becomes then

$$\sigma^2 = \frac{-(3\gamma-4)(\Omega_0+\mathcal{M}_0)}{I_0} . \tag{3.5.17}$$

Substituting this into the instability criterion equation (3.5.12), we have

$$(3\gamma-4)(\Omega_0+\mathcal{M}_0) < 0 . \tag{3.5.18}$$

Now, if we assume $\gamma > 4/3$, we have a sufficient condition for instability due to the presence of magnetic energy as follows:

$$\mathcal{M}_0 > |\Omega_0| . \tag{3.5.19}$$

In the following manner, we may obtain a crude estimate of the magnitude of the magnetic fields necessary to disrupt a star. The gravitational potential energy for a sphere of uniform density is

$$\Omega = -\frac{3}{5}\frac{GM^2}{R} , \tag{3.5.20}$$

while the magnetic energy is

$$\mathcal{M} = \frac{1}{8}\iiint |\mathbf{H}|^2 dx_1 dx_2 dx_3 = \frac{R^3\overline{|\mathbf{H}|^2}}{6} . \tag{3.5.21}$$

Combining equation (3.5.20) and equation (3.5.21), we see that the root mean square value of the magnetic field sufficient to disrupt a uniformly dense sphere is

$$\sqrt{\overline{|\mathbf{H}|^2}} > 2\times10^8 \frac{M}{R^2} \text{ gauss}, \tag{3.5.22}$$

where M and R are given in solar units. Thus, for a main sequence A star with $M = 4M_\odot$ and $R = 5R_\odot$, we have

$$H_{rms} > 3\text{x}10^7 \text{ gauss} . \tag{3.5.23}$$

However, for a star like VV Cephei with $M = 100\ M_\odot$, and $R = 2600\ R_\odot$, we have

$$H_{rms} > 3000 \text{ gauss} . \tag{3.5.24}$$

We may conclude from these arguments that for a main sequence star, an extremely large magnetic field would be sufficient to cause the star to become unstable. However, for an unusually large star, the required field becomes much smaller. In the case of VV Cephei, Babcock has measured a field ranging from +2000 to -1200 gauss. Thus, it would appear that VV Cephei is on the verge of being magnetically unstable. One might argue that our crude estimates of Ω are so crude as to be meaningless due to the large central concentration of the mass in giant stars. However, it should be pointed out that the magnetic field one can observe is, of necessity, a surface field and, therefore, provides us with a lower limit on the magnetic energy. Thus, we may have some hope that our limiting field values are not too far from being realistic.

It is interesting to note that the instability criterion equation (3.5.18) permits the existence of a gas with $\gamma < 4/3$ providing the magnetic energy exceeds the gravitational energy. Indeed, the stability criterion (3.5.18) would require a necessary condition for the stability of any configuration where $\mathcal{M} > |\Omega|$ that γ be less than 4/3. However, it is also true that the physical meaning of a gas having a $\gamma < 4/3$ is a little obscure.

If we now consider a rotating configuration with no magnetic field, equation (3.5.14) combined with the stability criterion equation (3.5.12) becomes

$$(5-3\gamma)\omega_0\mathcal{L}_0 > (3\gamma-4)\Omega_0 . \qquad 3.5.25$$

If we restrict γ to be less than 5/3 we have

$$\omega_0\mathcal{L}_0 \quad > \quad \frac{(3\gamma-4)\Omega_0}{(5-3\gamma)} \quad . \qquad 3.5.26$$

Since Ω_o is intrinsically negative, we see that the stability condition will always be satisfied with any ω_o. Thus, for all known stars the stability criterion for rotation is not particularly useful.

However, all this is not meant to imply that the rotational terms are unimportant. Indeed, Ledoux[1] has shown that rotational velocities encountered in stars may lead to a variation in the pulsational period by as much as 20%. Let us briefly consider the instability criterion when both magnetic and rotational energy are present. This may be obtained by combining equation (3.5.15) and

$$(3\gamma-4)(\Omega_0+\mathcal{M}_0)-(5-3\gamma)\omega_0\mathcal{L}_0 > 0 . \qquad 3.5.27$$

As before, this condition may never be satisfied unless $|\Omega_0| > \mathcal{M}_0$. However, even in the event that $|\Omega_0| > \mathcal{M}_0$, the condition may still not be satisfied because of the presence of the rotational term. Thus, it is evident that if $4/3 < \gamma < 5/3$, then the presence of rotation will help stabilize stars. This result is certainly not intuitive. A physical explanation of the result might be supplied by the following argument.

Consider a pulsating configuration containing both rotational and magnetic energy. As the system expands or contracts, a certain amount of energy will be required to slow down or speed up the rotation in order to keep the angular momentum constant. This energy must be supplied by the kinetic energy of the gas itself, and, since this is supplied by the potential energies present, ultimately must come from the gravitational and magnetic energies. Therefore, the amount of this energy transferred from the magnetic and gravitational energies will depend on γ. Also since the gravitational and magnetic energies must supply this energy to the rotation, the energy is no longer available to "feed" the pulsation and disrupt the star.

We may now ask what sort of increase in the maximum magnetic field can this additional rotational "stability" supply. From our previous investigation with the rotational stability criteria we might expect the result to be small. That is, since the rotational stability criteria did not supply us with as important a result as did the magnetic instability, we would expect the effects of rotation to be small compared with the magnetic energy. If one considers a uniform model with $\gamma = 3/2$, rotating at critical velocity, he will find the magnetic field may only be increased by about 0.3% before instability will again set in. Thus, even though stability is increased by the presence of rotation, it is not increased a great deal.

It is appropriate at this point to make some comments regarding all of the stability criteria relating to the stability of radial pulsations. It would have been more correct to employ the integral form of the expressions for the frequency of pulsation. However, the result one would obtain by using the integral expression and a specific model would only differ in degree from those derived here. It is hoped that the degree of differences would not be large. There is one respect in which the differences between the derived criteria and the 'correct' ones may result in a difference in kind. It must be remembered that the expressions developed for the pulsational frequencies were based on a first order theory as are the stability criteria developed in this section. However, the conditions at which one wishes to apply an instability criteria are generally such that the second, and higher, order terms are not small and should not be neglected. Chandrasekhar and Fermi[3] have shown that a sphere under the influence of a strong dipole field will tend to be "flattened" in much the same way as it will be by rotation. Once the spherical symmetry has been destroyed, either by the presence of a strong, magnetic field or rapid rotation, the concept of radial pulsation becomes inconsistent. As mentioned before, analysis of such systems would require the use of the tensor virial theorem and considerable insight into the types of perturbations to employ.

I would be remiss if I left the subject of the virial theorem and stability without some discussion of the recent questions raised with regard to the appropriateness of the approach. To me, these questions appear to be partly substantive and partly semantic and revolve largely around one of these modifiers mentioned earlier, namely, secular stability. So far, our discussion has been restricted to problems involving dynamical stability about which there seems to be little argument. A dynamically unstable system will disintegrate exponentially, usually on a time scale related to the hydrodynamical time scale for the system. Such destruction is usually so unambiguous that no complications arise in the use of the word unstable. Such is not the case for the term secular stability.

The notion of secular stability involves the response of the system to small dissipative forces, such as viscosity and thus must depend to some extent, on the nature of those forces. Time scales for development of instabilities will be governed by the forces and hence may be very long. Perhaps one of the clearest contemporary discussions of the term is given by Hunter[8] who notes that there is less than universal agreement on this meaning of the term. He points out that difficulties arise in rotating systems resulting from the presence of the coriolis forces, which lead to a clear distinction between dynamically and secularly stable systems. As we saw in Chapter II, section 5, the terms associated with the coriolis forces can be made to vanish by the proper choice of a coordinate frame and they would appear to play no role in the energy balance of the system. However, their variation does not vanish and hence they will affect the pulsational analysis. Since globally the forces are conservative the first result is not surprising and since radially moving mass in a rotating frame must respond to the conservation of angular momentum, neither is the second. Now if dissipative forces are present such as viscosity then it may be possible to redistribute local angular momentum while conserving it globally so that no equilibrium configuration is ever reached. In addition, except for the global constraint on the total angular momentum, no constraints are placed on the transfer of energy from the rotational field to the thermal field. Indeed the presence of dissipative forces guarantees that this must happen. Thus, instabilities associated with these forces might exist which would otherwise go undetected. This line of reasoning demonstrates a qualitative difference between the cases of uniform rotation and differential rotation in that in the former dissipative forces will be inactive and the analysis will be appropriate while in latter cases they must be explicitly included. This point is central to a lengthy series of papers[9, IO, 11, 12, 13] by Ostriker and others, which discuss the stability of a variety of differentially rotating systems. However, the majority of these papers clearly state that the authors are dealing with systems with zero viscosity and so the problem is not one of the accuracy of the analysis but rather of the applicability of the analysis to physical systems. In practice, the viscosity of the gas in most stars is so extremely low that the time scales for the development of instability arising from viscosity driven instabilities will be very long.

One cannot hope to untangle in a few paragraphs a controversy which has taken more than a decade to develop and at a formal mathematical level is quite subtle. However, it is worth noting that recent[14] statements which essentially say that the tensor virial approach to stability is wrong do nothing to clarify the situation. The presence of dissipative forces can be included in the equations of motion and thus in the resulting tensor representation of Lagrange's identity. The resulting stability analysis would then correctly reflect the presence of these forces and thus be dependent on their specific nature. Insofar as the time averaged form of Lagrange's identity, which is technically the virial theorem, is used the arguments of Milne[15] as presented in Chapter I still apply. The presence of velocity dependent forces does not affect the virial theorem unless those forces stop or destroy the system during the time over which the average is taken. At this level assailing the virial theorem is as useful an enterprise as denying the validity of a conservation law.

6. Summary

In this chapter we have explored the results of applying a specific analytical technique to the virial theorem. As in other chapters, we began with the simple and moved to the more complex. Having discussed the implications of the variational approach to the virial theorem we moved to develop the explicit form for the simple scalar theorem appropriate for self-gravitating systems. We recreated the pulsational formula [equation (3.2.19)] originally due to Ledoux. One implication of this result is that the fundamental mode of oscillation depends only on the square root of the density and when coupled with the stability criterion in section 5, leads immediately to the Jean's stability criterion. This is not surprising as both results have as the derivational origin the same concept (i.e., the equations of motion). However, it is reassuring when a different approach yields results already well accepted.

In section 3 we expanded the variational approach to include the effects of magnetic fields and rotation. In spite of many distractions dealing with the variation of magnetic fields, etc., the influence of these added features on the pulsation frequency and hence stability became clear. Rotation can either enhance or reduce the stability of a configuration depending on whether or not the value of γ for the gas permits net energy to be fed to the pulsation. The influence of an internal magnetic field is to destabilize the star for all realistic values of γ. However, the effect of a surface field proves to be more complex. Here the result depends critically on the geometry of the field. In the last section we dealt briefly with the overall question of stability and showed explicitly how the virial theorem provided an excellent basis for a linear stability analysis of a symmetric system. Throughout the chapter we confined ourselves to spherically symmetric systems exhibiting radial pulsations only. As mentioned, this is inappropriate when considering either rotation or magnetic fields as neither can exhibit spherical symmetry and thus one would expect non-radial oscillation to be excited. However, unless the field energies become quite large one would expect the pulsational frequencies not to differ greatly from the purely radial theory.

This line of reasoning becomes particularly dangerous when one turns to a discussion of stability. First the interesting situations of marginal stability are liable to involve substantial magnetic fields or rapid rotation. If these aspherical properties are large, the departure from spherical systems of the mass distribution will also be large invalidating every aspect of the analysis. In addition, for the stability analysis to be valid all possible modes of perturbation must be included. Limiting oneself to only the radial modes is to invite a misleading result. Fortunately the techniques for dealing with these problems exist and have been developed here as well as the literature. The tensor virial theorem as is presented in Chapter II, section 1 allows one to follow perturbations in independent spatial coordinates. In principle, a complete variational analysis of perturbations to all independent spatial coordinates will allow one to compute the non-radial as well as radial modes of oscillation and thereby obtain a much more secure analysis of the system's stability.

Notes to Chapter 3

3.1 Remember the defining expression for the gravitational potential energy is

$$\Omega = -G\int_0^M \frac{m(r)dm(r)}{r} \quad . \qquad \text{N3.1.1}$$

We may again use the fact that the variation of m(r) and dm(r) are both zero, to obtain

$$\delta\Omega = -G\int_0^M \frac{\delta r}{r^2} m(r)dm(r) , \qquad \text{N3.1.2}$$

which can be written as an energy integral by noting from equation (N3.1.1) that

$$\frac{d\Omega}{dm(r)} = -\frac{Gm(r)}{r} \quad . \qquad \text{N3.1.3}$$

Evaluating the above expression at $r = r_o$ and using the result in equation (3.2.11) to first order accuracy we get

$$\delta\Omega = -\int_0^{\Omega_0} \frac{\delta r}{r_0} d\Omega . \qquad \text{N3.1.4}$$

3.2 We may write the total energy as the sum of two energies $\mathcal{T}_1$ and $\mathcal{T}_2$, where $\mathcal{T}_1$ is the kinetic energy due to the mass motion of the gas arising from the pulsations themselves. Now the total kinetic energy of mass motion is given by

$$\mathcal{T}_1 = \tfrac{1}{2}\int_0^R 4\pi r^2 \rho \left(\frac{dr}{dt}\right)^2 dr = \tfrac{1}{2}\int_0^M \left(\frac{dr}{dt}\right)^2 dm(r) \quad . \qquad \text{N3.2.1}$$

Thus, the variation of $\mathcal{T}_1$

$$\delta\mathcal{T}_1 = \tfrac{1}{2}\int_0^M \left(\frac{dr}{dt}\right)\left(\frac{d\delta r}{dt}\right) dm(r) . \qquad \text{N3.2.2}$$

However, from the definition of δr we see that

$$r = r_o + \delta r . \qquad \text{N3.2.3}$$

Since the equilibrium point r_0 cannot vary with time by definition, equation (N3.2.2) becomes

$$\delta\mathcal{T}_1 = \tfrac{1}{2}\int_0^M \left(\frac{d(\delta r)}{dt}\right)^2 dm(r) . \qquad \text{N3.2.4}$$

The largest term in the integral of equation (N3.2.4) is second order in δr and may be neglected with respect to the first order terms of equation (3.2.13), and equation (3.2.6). Thus, to first order we have

$$\delta T = \delta\mathcal{T}_1 + \delta\mathcal{T}_2 \cong \delta\mathcal{T}_2 \quad . \qquad \text{N3.2.5}$$

Now consider the kinetic energy of a small volume of an ideal gas.

$$d\mathcal{T}_2 = \tfrac{3}{2} Nk\mathcal{T}dV . \qquad \text{N3.2.6}$$

However, the gas pressure is given by $P_g = NkT$ and $dm(r) = \rho dV$. Therefore,

$$d\mathcal{T}_2 = \tfrac{3}{2}\frac{P_g}{\rho}dm(r) . \qquad \text{N3.2.7}$$

Thus, twice the total kinetic energy of the gas sphere arising from thermal sources is

$$2\mathcal{T}_2 = 3\int_0^M \frac{P_g}{\rho}dm(r) . \qquad \text{N3.2.8}$$

Now neglecting radiation pressure, so that the total pressure is equal to the gas pressure, and remembering that the variation of dm(r) is zero, we have

$$2\delta T = 2\delta\mathcal{T}_2 = 3\int_0^M \delta\left(\frac{P_g}{\rho}\right)dm(r) . \qquad \text{N3.2.9}$$

We shall now assume that the pulsations are adiabatic so that

$$\frac{\delta P}{P} = \gamma\frac{\delta\rho}{\rho} , \qquad \text{N3.2.10}$$

where γ is the ratio of specific heats c_p / c_v . Now

$$\delta\left(\frac{P}{\rho}\right) = \frac{\rho\delta P - P\delta\rho}{\rho^2} = \frac{\left(\delta P/P\right)\rho - \delta\rho}{\left(\rho^2/P\right)} = \frac{P}{\rho^2[(\gamma-1)\delta\rho_0]} . \qquad \text{N3.2.11}$$

Therefore, again evaluating at the equilibrium position, substituting into equation (N3.2.9),
And keeping only terms up to first order, we have

$$2\delta T \cong 3\int_0^M \frac{P_0}{\rho_0}(\gamma-1)\frac{\delta\rho}{\rho_0}dm(r) . \qquad \text{N3.2.12}$$

3.3 In out attempt to obtain an expression for $\delta\rho/\rho$and thereby determining the variation energy, we shall invoke the following argument. From the conservation of mass, we have

$$\delta m(r') = 0 = \int_0^{r'} 4\pi r^2\rho dr = \int_0^{r'} 4\pi(2r\delta r)\rho dr + \int_0^{r'} 4\pi r^2\delta\rho dr + \int_0^{r'} 4\pi r^2\rho d(\delta r) . \qquad \text{N3.3.1}$$

Rewriting equation (3.2.28) and evaluating at $r = r_0$, we have

$$\int_0^{r'} r_0^2\delta\rho dr_0 = -\int_0^{r'} \rho_0 r_0^2\left[\frac{2\delta r dr_0}{r_0} + d(\delta r)\right] . \qquad \text{N3.3.2}$$

From the definition of ξ (equation 3.2.12), we have

$$\frac{d\xi}{dr_0} = \frac{r_0 d(\delta r)/dr - dr}{r_0^2} \text{ or, } d\xi = \frac{d(\delta r)}{r_0} - \frac{\delta r dr_0}{r_0^2} . \qquad \text{N3.3.3}$$

Eliminating d(δr) from equation (N3.3.2) with the aid of equation (N3.3.3), we have again to the first

$$\int_0^{r'} r_0^2\delta\rho dr_0 = -\int_0^{r'}\rho_0 r_0^2\left[3 + r_0\frac{d\xi}{dr_0}\right]dr_0 \; . \qquad \text{N3.3.4}$$

Equation (N3.3.4) must hold for all values of r'. This can only be true if the integrands equal. Thus,

$$\frac{\delta\rho}{\rho_0} = -\left(3\xi + r_0\frac{d\xi}{dr_0}\right) \; . \qquad \text{N3.3.5}$$

3.4 In an attempt to simplify equation (3.2.15), let us consider the second integral.

$$3\int_0^M \frac{P_0}{\rho_0}(\gamma-1)r_0\frac{d\xi_0}{dr_0}e^{i\sigma t}dm(r) = 3\int_0^R \frac{P_0}{\rho_0}(\gamma-1)r_0\frac{d\xi_0}{dr_0}e^{i\sigma t}4\pi r_0^2 dr_0 \; . \qquad \text{N3.4.1}$$

Integrating the right hand side by parts we obtain

$$12\pi e^{i\sigma t}\int_0^{R_0}\left(\frac{d\xi_0}{dr_0}\right)\frac{P_0}{\rho_0}(\gamma-1)r_0^3 dr_0 = 12\pi e^{i\sigma t}P_0(\gamma-1)r_0^3\xi_0\Big|_0^R - 12\pi e^{i\sigma t}\int_0^R \xi_0\frac{d}{dr}\left[P_0(\gamma-1)r_0^3\right]dr_0 \; . \qquad \text{N3.4.2}$$

At this point we shall impose the boundary conditions that $P_o \rightarrow 0$ at $r_o = R$ and $\xi_o \rightarrow 0$ at $r_o = 0$. The first of these conditions is the familiar condition of stellar structure which essentially defines the surface of the gas sphere. The second condition is required by assuming that the radial pulsation is a continuous function. With these conditions, the integrated part of equation (N3.4.2) vanishes and the remaining integral becomes:

$$\int_0^R \xi_0\frac{d}{dr}\left[P_0(\gamma-1)r_0^3\right]dr_0 = \int_0^R P_0\xi_0 r_0^3 + \int_0^R(\gamma-1)\xi_0\frac{dP_0}{dr_0}r_0^3 dr_0 + 3\int_0^R P_0(\gamma-1)\xi_0 r_0^2 dr_0 \; . \qquad \text{N3.4.3}$$

Conservation of momentum (i.e., hydrostatic equilibrium) requires that

$$\frac{dP_0}{dr_0} = -\frac{Gm(r_0)\rho}{r_0^2} \; . \qquad \text{N3.4.4}$$

Therefore,

$$4\pi r_0^3\frac{dP_0}{dr_0} = -\frac{4\pi G r_0^2 m(r_0)\rho}{r_0} = -\frac{Gm(r_0)}{r_0}\frac{dm(r_0)}{dr_0} = \frac{d\Omega_0}{dr_0} \; . \qquad \text{N3.4.5}$$

Making use of the second form of equation (N3.4.5) to simplify the second integral in equation (N3.4.3), and the definition of dm(r) to simplify the other two integrals in equation (N3.4.3), we obtain the following expression for the variation of the kinetic energy [i.e. equation (3.2.15)].

$$\begin{aligned}2\delta T \cong & -9e^{i\sigma t}\int_0^M \frac{P_0\xi_0}{\rho_0}(\gamma-1)dm(r_0) + 3e^{i\sigma t}\int_0^M \frac{P_0\xi_0 r_0}{\rho_0}\frac{d\gamma}{dr_0}dm(r_0) + 3e^{i\sigma t}\int_0^{\Omega_0}\xi_0(\gamma-1)d\Omega_0 \\ & + 9e^{i\sigma t}\int_0^M \frac{P_0\xi_0}{\rho_0}(\gamma-1)dm(r_0)\end{aligned} \; . \qquad \text{N3.4.6}$$

The above expression is obtained by making use of the previously mentioned definition and substituting it into equation (N3.4.3), then into equation (N3.4.2), and finally into equation (3.2.15). Since the first and last integrals are identical except for the difference in sign, they vanish from the expression and

$$2\delta T \cong 3e^{i\sigma t}\int_0^M \frac{P_0\xi_0 r_0}{\rho_0}\frac{d\gamma}{dr_0}dm(r_0) + 3e^{i\sigma t}\int_0^{\Omega_0}\xi_0(\gamma-1)d\Omega_0 \quad . \qquad \text{N3.4.7}$$

3.5 Let us define

$$\eta_i \equiv \delta x_i \, . \qquad \text{N3.5.1}$$

Now the definition of the mass within a given volume in Cartesian coordinates becomes

$$m(V) = \iiint \rho dx_1 dx_2 dx_3 \, . \qquad \text{N3.5.2}$$

The conservation of mass requires, as it did in section 2, that the variation of the mass is zero. Thus taking the variations of equation (N3.5.2), we obtain

$$\delta m(V) = \iiint \delta\rho dx_1 dx_2 dx_3 + \iiint \rho d(\delta x_1)dx_2 dx_3 + \iiint \rho dx_1 d(\delta x_2)dx_3 \iiint \rho dx_1 dx_2 d(\delta x_3) \, . \qquad \text{N3.5.3}$$

Rewriting the last three integrals, we have

$$\iiint \delta\rho dx_1 dx_2 dx_3 = -\iiint \rho \sum_{i=1}^{3}\frac{d(\delta x_i)}{dx_i}dx_1 dx_2 dx_3 \; . \qquad \text{N3.5.4}$$

Since $\delta x_i = \eta_i$, we have by the chain rule

$$\frac{d(\delta x_i)}{dx_i} = \frac{d\eta_i}{dx_i} = \sum_{j=1}^{3}\frac{dx_i}{dx_j}\frac{\partial\eta_i}{\partial x_j} \quad , \qquad \text{N3.5.5}$$

However, since the x_i's are linearly independent and the η_i's are just the variation of these coordinates, not only is the second term in the product [under the summation sign of equation (N3.5.5)] zero if i= j but, so is the first term.

Thus as might be expected from the orthogonality of the x_i's we have

$$\frac{d\eta_i}{dx_i} = \frac{\partial\eta_i}{\partial x_i} \quad . \qquad \text{N3.5.6}$$

Substitution of this into equation (N3.5.4), yields

$$\iiint \delta\rho dx_1 dx_2 dx_3 = -\iiint \rho \sum_{i=1}^{3}\frac{\partial\eta_i}{\partial x_i}dx_1 dx_2 dx_3 \, . \qquad \text{N3.5.7}$$

Equation (N3.5.7) must hold for any volume of integration. This can only be true if the integrands themselves are equal. Thus, we finally obtain

$$\frac{\delta\rho}{\rho} = -\sum_{i=1}^{3}\frac{\partial\eta_i}{\partial x_i} \, . \qquad \text{N3.5.8}$$

3.6 Suppose a displacement ***n*** takes place with the slow continuous movement so that

$$\boldsymbol{u} = \frac{d\boldsymbol{n}}{dt} . \qquad \text{N3.6.1}$$

Then, if the electrical conductivity of the medium is infinite, the time-variation of the electric field is

$$\Delta\mathbf{E} = -\boldsymbol{u}\times\mathbf{H} \quad . \qquad \text{N3.6.2}$$

However, Maxwell's equations for an infinitely conducting medium require that

$$\nabla\times(\Delta\mathbf{E}) = -\frac{\partial}{\partial t}\Delta\mathbf{H} \quad . \qquad \text{N3.6.3}$$

Combining equation (N3.6.2) and equation (N3.6.3), we have

$$-\frac{\partial(\Delta\mathbf{H})}{\partial t} = \nabla\times(-\boldsymbol{u}\times\mathbf{H}) \; . \qquad \text{N3.6.4}$$

The integral form of equation (N3.6.4) is just

$$\Delta\mathbf{H} = \nabla\times(\boldsymbol{n}\times\mathbf{H}) . \qquad \text{N3.6.5}$$

But, from the definition of the time and space variations and the equation relating to the total time derivatives, we know that

$$\Delta\mathbf{H} = \frac{\partial\mathbf{H}}{\partial t}dt , \qquad \text{N3.6.6}$$

and

$$\frac{\partial}{\partial t} = \frac{d}{dt} - \mathbf{v}\bullet\nabla , \qquad \text{N3.6.7}$$

which results in

$$\Delta\mathbf{H} = \left(\frac{d\mathbf{H}}{dt} - \sum_j \frac{d\mathbf{x}_j}{dt}\bullet\nabla\mathbf{H}\right)dt \quad . \qquad \text{N3.6.8}$$

However,

$$\frac{d}{dt} = \sum_i \frac{\partial}{\partial x_i}\frac{dx_i}{dt} , \qquad \text{N3.6.9}$$

while definition of the space variation δ**H** is

$$\delta\mathbf{H} = \sum_i \frac{\partial\mathbf{H}}{\partial x}dt\bullet\nabla\mathbf{H} . \qquad \text{N3.6.10}$$

Combining equations (N3.6.l0), (N3.6.9), and equation (N3.6.8), we have

$$\Delta\mathbf{H} = \delta\mathbf{H} - \sum_i \frac{d\mathbf{x}_i}{dt}dt\bullet\nabla\mathbf{H} \quad . \qquad \text{N3.6.11}$$

Noting that the variation of a linearly independent quantity may be interpreted as the total differential of the quantity

$$dx_i = \sum_i \frac{\partial x_i}{\partial x_j}dx_j = \delta x_i , \qquad \text{N3.6.12}$$

we obtain

$$\Delta\mathbf{H} = \delta\mathbf{H} - \boldsymbol{n}\bullet\nabla\mathbf{H} \quad . \qquad \text{3.6.13}$$

Combining equation (N3.6.l3) and equation (N3.6.S), we finally obtain an equation for the variation of the magnetic field ($\delta\mathbf{H}$).

$$\delta\mathbf{H} = \nabla\times(\boldsymbol{n}\times\mathbf{H}) - (\boldsymbol{n}\bullet\nabla)\mathbf{H} \quad . \qquad \text{N3.6.14}$$

Now the curl of a cross-product may be written as

$$\nabla\times(\boldsymbol{n}\times\mathbf{H}) = (\mathbf{H}\bullet\nabla)\boldsymbol{n} - \mathbf{H}(\nabla\bullet\boldsymbol{n}) - (\boldsymbol{n}\bullet\nabla)\mathbf{H} + \boldsymbol{n}(\nabla\bullet\mathbf{H}) \quad . \qquad \text{N3.6.15}$$

Since the divergence of **H** is always zero, the last term vanishes. Combining this expression for the curl of a cross-product with equation (N3.6.l4) yields

$$\delta\mathbf{H} = (\mathbf{H}\bullet\nabla)\boldsymbol{n} - \mathbf{H}(\nabla\bullet\boldsymbol{n}) - (\boldsymbol{n}\bullet\nabla)\mathbf{H} + (\boldsymbol{n}\bullet\nabla)\mathbf{H} \,. \qquad \text{N3.6.16}$$

The last two terms are identical except for opposite signs and thus cancel out. The remaining expression may be written in component form as follows:

$$\delta H_i = \sum_j \left(H_j \frac{\partial\eta_i}{\partial x_j} - H_i \frac{\partial\eta_j}{\partial x_i} \right) \quad . \qquad \text{N3.6.17}$$

3.7 In the section 3.2 we went through an extensive argument (see Note *3.6*, equation N3.6.l6), to show that

$$\delta\mathbf{H} = (\mathbf{H}\bullet\nabla)\,\delta\mathbf{r} - \mathbf{H}(\nabla\bullet\delta\mathbf{r})\,, \qquad \text{N3.7.1}$$

where δr plays the role of η in that discussion. The easiest way to evaluate the first term is in Cartesian coordinates, remembering that ξ_0 is constant. Then,

$$(\mathbf{H}\bullet\nabla)\delta\mathbf{r} = \sum_i H_i \frac{\partial(\xi x_i)}{\partial x_i} = \xi\mathbf{H} \,. \qquad \text{N3.7.2}$$

The second term can be evaluated the same way, so that

$$\mathbf{H}(\nabla\bullet\delta\mathbf{r}) = H_j \sum_i \frac{\partial(\xi x_i)}{\partial x_i} = 3\xi\mathbf{H} \,. \qquad \text{N3.7.3}$$

So, the variation of the magnetic field has taken the particularly simple form

$$\delta\mathbf{H} = -2\xi\mathbf{H}_0 \quad . \qquad \text{N3.7.4}$$

References

1 Ledoux, P. (1945), Ap. J. 102, 143.
(1958), *Handbuch der Physik*, Springer-Ver1sg, Berlin, Gottingen, Heide1berg, pp. 605-687.
2 Chandrasekhar, S. (1963), Ap. J. 137, p. ll85.
3 Chandrasekhar, S. and Fermi, E. (1953), Ap. J. ll8, p. ll6.
4 Ledoux, P. and Walraven, (1958), *Handbook der Physik*, Vol. 51, pp. 431-592, Springer-Ver1ag, Berlin, Gottingen, Heidelberg.
5 Chandrasekhar, S. and Limber, D. N. (1959), Ap. J. ll9, p. 10.
6 Chandrasekhar, S. (1961), *Hydrodynamic and Hydromagnetic Stability*, Oxford University Press, London, p.
7 . (1939), *An Introduction to the Study of Stellar Structure*, Dover Pub., Inc. (1957), p. 53.
8 Hunter, C. (1977), Ap. J. 213, p.497.
9 Lyden-Bell, D. and Ostriker, J. P. (1967), M.N.R.A.S. 136, p. 293.
10 Tassoul, J. L. and Ostriker, J. P. (1968), Ap. J. 154, p. 613.
II Ostriker, J. P. and Tassoul, J. L. (1969), Ap. J. 155, p. 987.
12 Ostriker, J. P. and Bodenheimer (1973), Ap. J. 180, p. 171.
13 Ostriker, J. P. and Peebles, P. J. E. (1973), Ap. J. 186, p. 467.
14 Bardeen, J. M., Friedman, J. L., Schutz, B. F. and Sarkin, R. (1977), Ap. J. Lett. 217, p. L49.
15 Milne, E. A. (1925), Phil. Mag. S. 6, Vol. 50, pp. 409-419.