4

Potential Theory

We have seen how the solution of any classical mechanics problem is first one of determining the equations of motion. These then must be solved in order to find the motion of the particles that comprise the mechanical system. In the previous chapter, we developed the formalisms of Lagrange and Hamilton, which enable the equations of motion to be written down as either a set of n second order differential equations or 2n first order differential equations depending on whether one chooses the formalism of Lagrange or Hamilton. However, in the methods developed, the Hamiltonian required knowledge of the Lagrangian, and the correct formulation of the Lagrangian required knowledge of the potential through which the system of particles moves. Thus, the development of the equations of motion has been reduced to the determination of the potential; the rest is manipulation. In this way the more complicated vector equations of motion can be obtained from the far simpler concept of the scalar field of the potential.

To complete this development we shall see how the potential resulting from the sources of the forces that drive the system can be determined. In keeping with the celestial mechanics theme we shall restrict ourselves to the forces of gravitation although much of the formalism had its origins in the theory of electromagnetism - specifically electrostatics. The most notable difference between gravitation and electromagnetism (other than the obvious difference in the strength of the force) is that the sources of the gravitational force all have the same sign, but all masses behave as if they were attractive.

4.1 The Scalar Potential Field and the Gravitational Field

In the last chapter we saw that any forces with zero curl could be derived from a potential so that if

$$\nabla \times \vec{F} = 0 \quad , \tag{4.1.1}$$

then

$$\vec{F} = \nabla V \quad , \tag{4.1.2}$$

where V is the potential energy. Forces that satisfy this condition were said to be conservative so that the total energy of the system was constant. Such is the case with the gravitational force. Let us define the gravitational potential energy as Ω so that the gravitational force will be

$$\vec{F} = -\nabla \Omega \quad . \tag{4.1.3}$$

Now by analogy with the electromagnetic force, let us define the gravitational field $\vec{G}$ as the gravitational force per unit mass so that

$$\vec{G} = \vec{F}/m = -(\nabla \Omega / m) \equiv \nabla \Phi \quad . \tag{4.1.4}$$

Here Φ is known as the gravitational potential, and from the form of equation (4.1.4) we can draw a direct comparison to electrostatics. $\vec{G}$ is analogous to the electric field while Φ is analogous to the electric potential.

Now Newtonian gravity says that the gravitational force between any two objects is proportional to the product of their masses and inversely proportional to the square of the distance separating them and acts along the line joining them. Thus the collective sum of the forces acting on a particle of mass m will be

$$\vec{F}_g(r) = \sum_i \frac{GmM_i(\vec{r}_i - \vec{r})}{|\vec{r}_i - \vec{r}|^3} = \int_{V'} \frac{Gm\rho(\vec{r}_i - \vec{r})}{|\vec{r}_i - \vec{r}|^3} dV' \quad , \tag{4.1.5}$$

where we have included an expression on the right to indicate the total force arising from a continuous mass distribution ρ(r). Thus the gravitational field resulting from such a configuration is

$$\vec{G}_g(r) = \sum_i \frac{GM_i(\vec{r}_i - \vec{r})}{|\vec{r}_i - \vec{r}|^3} = \int_{V'} \frac{G\rho(\vec{r}_i - \vec{r})}{|\vec{r}_i - \vec{r}|^3} dV' \quad . \tag{4.1.6}$$

The potential that will give rise to this force field is

$$\Phi(\vec{r}) = \sum_i \frac{GM_i}{|\vec{r}_i - \vec{r}|} = \int_{V'} \frac{G\rho(\vec{r}\,')}{|\vec{r}\,' - \vec{r}|} dV' \quad . \tag{4.1.7}$$

The evaluation of the scalar integral of equation (4.1.7) will provide us with the potential (and hence the potential energy of a unit mass) ready for insertion in the Lagrangian. In general, however, such integrals are difficult to do so we will consider a different representation of the potential in the hope of finding another means for its determination.

4.2 Poisson's and Laplace's Equations

The basic approach in this section will be to turn the integral expression for the potential into a differential expression in the hope that the large body of knowledge developed for differential equations will enable us to find an expression for the potential. To do this we will have to make a clear distinction between the coordinate points that describe the location at which the potential is being measured (the field point) and the coordinates that describe the location of the sources of the field (source points). It is the latter coordinates that are summed or integrated over in order to obtain the total contribution to the potential from all its sources. In equations (4.1.5 - 4.1.7) $\vec{r}$ denotes the field points while $\vec{r}\,'$ labels the sources of the potential.

Consider the Laplacian {i.e. the divergence of the gradient [$\nabla^2 = (\nabla \bullet \nabla)$]} operating on the integral definition of the potential for continuous sources [the right-most term in equation (4.1.7)]. Since the Laplacian is operating on the potential, we really mean that it is operating on the field coordinates. But the field and source coordinates are independent so that we may move the Laplacian operator through the integral sign in the potential's definition. Thus,

$$\nabla^2 \Phi(\vec{r}) = \int_{V'} G\rho(\vec{r}\,') \left(\nabla \bullet \nabla \left[\frac{1}{|\vec{r}\,' - \vec{r}|} \right] \right) dV' \quad . \tag{4.2.1}$$

Since the Laplacian is the divergence of the gradient we may make use of the Divergence theorem

$$\int_V \nabla \bullet \vec{H} dv = \int_S \vec{H} \bullet d\vec{A} \quad , \tag{4.2.2}$$

to write

$$\int_{V'} G\rho(\vec{r}\,')[\nabla \bullet \nabla(|\vec{r}\,'-\vec{r}|^{-1})]dV' = \int_A G\rho(\vec{r}\,')[\nabla(|\vec{r}\,'-\vec{r}|^{-1})] \bullet d\vec{A} \quad . \qquad (4.2.3)$$

Here the surface A is that surface that encloses the volume V'.

Now consider the simpler function (l/r) and its gradient so that

$$\int_A \nabla\left[\tfrac{1}{r}\right] \bullet d\vec{A} = -\int_A \frac{\hat{r} \bullet d\vec{A}}{r^2} = -\int_A d\omega = -\omega \quad . \qquad (4.2.4)$$

The integrand of the second integral is just the definition of the differential solid angle so that the integral is just the solid angle ω subtended by the surface A as seen from the origin of r. If the source and field points are different physical points in space, then we may construct a volume that encloses all the source points but does not include the field point. Since the field point is outside of that volume, then the solid angle of the enclosing volume as seen from the field point is zero. However, should one of the source points correspond to the field point, the field point will be completely enclosed by the surrounding volume and the solid angle of the surface as seen from the field point will be 4π steradians. Therefore the integral on the right hand side of equation (4.2.3) will either be finite or zero depending on whether or not the field point is also a source point. Integrands that have this property can be written in terms of a function known as the Dirac delta function which is defined as follows:

$$\left.\begin{array}{l} \delta(r) \equiv 0 \quad \forall\ r \neq 0 \\ \int \delta(r)dr \equiv 1 \end{array}\right\} \quad . \qquad (4.2.5)$$

If we use this notation to describe the Laplacian of (l/r) we would write

$$\nabla^2(1/r) = -4\pi\delta(r) \quad , \qquad (4.2.6)$$

and our expression for the potential would become

$$\nabla^2\Phi(r) = \nabla^2\int_{V'} G\rho(\vec{r}\,')(|\vec{r}\,'-\vec{r}|)^{-1}dV' = -4\pi G\int_{V'}\delta(r'-r)\rho(\vec{r}\,')dV' \quad . \qquad (4.2.7)$$

This integral has exactly two possible results. If the field point is a source point we get

$$\nabla^2\Phi(r) = -4\pi G\rho(r) \quad , \tag{4.2.8}$$

which is known as *Poisson's equation*. If the field point is not a source point, then the integral is zero and we get

$$\nabla^2\Phi(r) = 0 \quad . \tag{4.2.9}$$

This is known as *Laplace's equation* and the solution of either yields the potential required for the Lagrangian and the equations of motion. Entire books have been written on the solution of these equations and a good deal of time is spent in the theory of electrostatics developing such solutions (eg. Jackson[4]). All of that expertise may be borrowed directly for the solution of the potential problem for mechanics.

In celestial mechanics we are usually interested in the motion of some object such as a planet, asteroid, or spacecraft that does not contribute significantly to the potential field in which it moves. Such a particle is usually called a test particle. Thus, it is Laplace's equation that is of the most interest. Laplace's equation is a second order partial differential equation. The solution of partial differential equations requires "functions of integration" rather than constants of integration expected for total differential equations. These functions are known as boundary conditions and their functional nature greatly complicates the solution of partial differential equations. The usual approach to the problem is to find some coordinate system wherein the functional boundary conditions are themselves constants. Under these conditions the partial differential equations in the coordinate variables can be written as the product of total differential equations, which may be solved separately. Such coordinate systems are said to be coordinate systems in which Laplace's equation is separable. It can be shown that there are thirteen orthonormal coordinate frames (see Morse and Feshback[1]) in which this can happen. Unless the boundary conditions of the problem are such that they conform to one of these coordinate systems, so that the functional conditions are indeed constant on the coordinate axes, one must usually resort to numerical methods for the solution of Laplace's equation.

Laplace's equation is simply the homogeneous form of Poisson's equation. Thus, any solution of Poisson's equation must begin with the solution of Laplace's equation. Having found the homogeneous solution, one proceeds to search for a particular solution. The sum of the two then provides the complete solution for the inhomogeneous Poisson's equation.

In this book we will be largely concerned with the motion of objects in the solar system where the dominant source of the gravitational potential is the sun (or some planet if one is discussing satellites). It is generally a good first approximation to assume that the potential of the sun and planets is that of a point mass. This greatly facilitates the solution of Laplace's equation and the determination of the potential. However, if one is interested in the motion of satellites about some non-spherical object then the situation is rather more complicated. For the precision required in the calculation of the orbits of spacecraft, one cannot usually I assume that the driving potential is that of a point mass and therefore spherically symmetric. Thus, we will spend a little time investigating an alternative method for determining the potential for slightly distorted objects.

4.3 Multipole Expansion of the Potential

Let us return to the integral representation of the gravitational potential

$$\Phi(r) = G\int_{V'} \frac{\rho(r')dV'}{|\vec{r}'-\vec{r}|} \quad . \qquad (4.3.1)$$

Assume that the motion of the test particle is such that it never comes "too near" the sources of the potential so that $|\vec{r}'| << |\vec{r}|$. Then we may expand the denominator of the integrand of equation (4.3.1) in a Taylor series about r' so that

$$\frac{1}{|r'-r|} \approx \frac{1}{r} + \sum_i r_i'\frac{\partial(1/r)}{\partial r_i} + \frac{1}{2}\sum_i\sum_j r_i'r_j'\frac{\partial^2(1/r)}{\partial r_i\partial r_j} - \frac{1}{6}\sum_i\sum_j\sum_k r_i'r_j'r_k'\frac{\partial^3(1/r)}{\partial r_i\partial r_j\partial r_k} \ . \qquad (4.3.2)$$

or in vector notation

$$[|\vec{r}'-\vec{r}|]^{-1} = (1/r) - [\vec{r}'\bullet\nabla(1/r)] + \tfrac{1}{2}[\vec{r}'\vec{r}'\colon\nabla\nabla(1/r)] - \frac{1}{6}[\vec{r}'\vec{r}'\vec{r}'\therefore\nabla\nabla\nabla(1/r)] \ . \qquad (4.3.3)$$

In Chapter 1 we defined the scalar product to represent complete summation over all available indices so that the resulting scalar product of tensors with ranks m and n was $|m-n|$. However, in order to make clear that multiple summations are needed in equation (4.3.3), I have used multiple "dots". The definition of this notation can be seen from the explicit summation in equation (4.3.2) or can be defined by

$$\vec{A}\vec{B}\colon\vec{A}'\vec{B}' = (\vec{A}\bullet\vec{B}')(\vec{B}\bullet\vec{A}') \quad . \qquad (4.3.4)$$

Using this expansion to replace the denominator of the integral definition of the potential [equation (4.3.1)] we get

$$\left.\begin{aligned}\Phi(r) = G\int_{V'}\frac{\rho(r')dV'}{|\vec{r}'-\vec{r}|} = G\Bigg[\frac{1}{r}\int_{V'}\rho(r')dv' - \int_{V'}\vec{r}'\rho(r')dV'\bullet\nabla\left(\frac{1}{r}\right) \\ +\tfrac{1}{2}\int_{V'}\vec{r}'\vec{r}'\rho(r')dV' \mathbin{:} \nabla\nabla\left(\frac{1}{r}\right)+\cdots+\Bigg]\end{aligned}\right\} . \quad (4.3.5)$$

This expansion allows the separation of the dependence of the field coordinates from the source coordinate. Thus the integrals are properties of the source of the potential only and may be calculated separately from any other aspect of the mechanics problem. Once known, they give the potential explicitly as a function of the field coordinates alone and this is what we need for specifying the Lagrangian. We can make this clearer by re-writing equation (4.3.5) as

$$\Phi(r) = \left\{ M(1/r) - \vec{P}\bullet\nabla(1/r) + \tfrac{1}{2}\mathbf{Q}\mathbin{:}\nabla\nabla(1/r) - \frac{1}{6}\overline{\mathbf{S}} \because \nabla\nabla\nabla(1/r) + \cdots + \right\} . \quad (4.3.6)$$

This expansion of the potential is known as a "multipole" expansion for the parameters $M, \vec{P}$, $\mathbf{Q}$, and $\overline{\mathbf{S}}$ which are known as the multipole moments of the source distribution. For the gravitational potential the unipole moment is a scalar and just equal to the total mass of the sources of the potential. The vector quantity $\vec{P}$ is called the dipole moment and $\mathbf{Q}$ is the tensor quadrupole moment, etc. The higher order moments are in turn higher order tensors. The repeated operation of the del-operator ∇ on the quantity (l/r) also produces higher order tensors, which are simply geometry and have nothing to do with the mass distribution itself. The first two of these are

$$\left.\begin{aligned}\nabla(1/r) &= -\hat{r}/r^2 \\ \nabla\nabla(1/r) &= -(\mathbf{1} - 3\hat{r}\hat{r})/r^3\end{aligned}\right\} . \quad (4.3.7)$$

As one considers higher order terms the geometrical tensors represented by the multiple gradient operators contain a larger and larger inverse dependence on r and therefore play a successively diminished role in determining the potential. Thus we have effectively separated the positional dependence of the field point from the mass distribution that produces the various multipole moments.

By way of example, let us consider two *unequal* mass points separated by a distance l, located on the z-axis, and with the coordinate origin at the center of mass (see Figure 4.1). From the definition of the multipole moments, we have

$$\left.\begin{aligned}
&M \equiv \int_{V'} \rho(r')dV' = m_1 + m_2 \\
&\vec{P} \equiv \int_{V'} \vec{r}'\rho(r')dV' = \left[\frac{m_1 m_2}{(m_1+m_2)} - \frac{m_2 m_1}{(m_1+m_2)}\right] l\,\hat{k} = 0 \\
&\mathbf{Q} \equiv \int_{V'} \vec{r}'\vec{r}'\rho(r')dV' = \left[\frac{m_1 m_2}{(m_1+m_2)}\right] l^2\,\hat{k}\hat{k} \\
&\overline{\mathbf{S}} \equiv \int_{V'} \vec{r}'\vec{r}'\vec{r}'\rho(r')dV' = \left(m_1 z_1^3 + m_2 z_2^3\right)\hat{k}\hat{k}\hat{k}
\end{aligned}\right\} , \qquad (4.3.8)$$

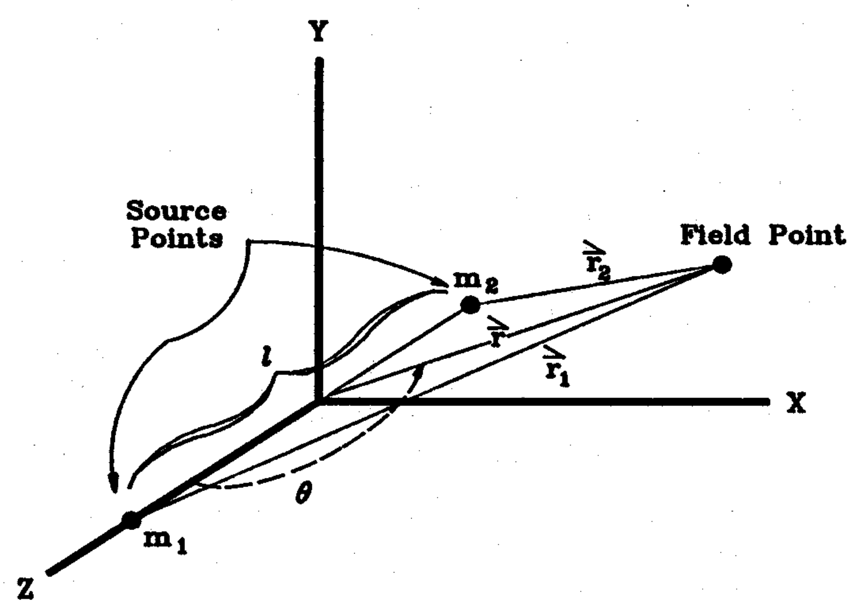

Figure 4.1 shows the arrangement of two unequal masses for the calculation of the multipole potential resulting from them.

which, when combined with the coordinate representation of equation (4.3.6), yields a series expansion for the potential of the form

$$\Phi(r) = G\left\{\frac{(m_1+m_2)}{r} + \left(\frac{l^2}{2}\right)\left[\frac{m_1 m_2}{(m_1+m_2)}\right]\frac{\left(1-3\cos^2\theta\right)}{r^3} + \cdots + \right\} \quad . \tag{4.3.9}$$

Unless the field point comes particularly close to the sources, this series will converge quickly. We can also make use of a pleasant property of the gravitational force, namely that there are no negative "charges" in the force law of gravitation. Thus we may always choose a coordinate system such that

$$\vec{P}(\vec{r}\,') = \int_{V'} \vec{r}\,'\rho(r')dV' = 0 \quad . \tag{4.3.10}$$

This means that for celestial mechanics there will never be a dipole moment of the potential as long as we choose the coordinate frame properly. This is usually done by taking advantage of any symmetry presented by the object and locating the origin at the center of mass. Not only does the dipole moment vanish, but for objects exhibiting plane symmetry all odd moments of the multipole expansion vanish for the gravitational potential. This certainly enhances the convergence of the series expansion for the potential and means that the first term that must be included after the point-mass potential term is the quadrapole term. The inclusion of this term means that the error in the potential will be of the order $O(1/r^5)$. Even though the potential represented by a multipole expansion converges rapidly with increasing distance, the contribution of such terms can be significant for small values of r. Thus there is great interest in determining the magnitude of these terms for the potential field of the earth so that the orbits of satellites may be predicted with greater certainty.

We have now described methods whereby the potential can be calculated for an arbitrary collection of mass points to an arbitrary degree of accuracy. The insertion of the potential into the Lagrangian will enable one to determine the equations of motion and the solution of these equations then constitutes the solution of any classical mechanics problem. Therefore, let us now turn to the solution of specific problems found in celestial mechanics.

Chapter 4: Exercises

1. The potential energy of the interaction between a multipole **T** and a scalar potential field Φ is given by

$$U = \mathbf{T}^{(i)} \odot \nabla^{(i)} \Phi \ ,$$

where $\mathbf{T}^{(i)}$ is a tensor of rank (i) and $\nabla^{(i)}\Phi$ describes i applications of the del operator to the scalar potential Φ. The symbol $\odot$ stands for the most general application of the scalar product, namely the contraction (i.e., the addition) of the two resulting tensors over all indices.

a: Consider four equal masses with Cartesian three dimensional coordinates

mass #	X	Y	Z
#1	-1	0	0
#2	+1	0	0
#3	0	2	2
#4	0	-3	-3

Find the total self energy of the system.

b: Find the potential energy of the above system with a fifth identical mass located at (0,0,0).

2. Given that the interaction energy of a dipole and quadrupole may be written as

$$\left(\begin{array}{l} U_{pq} = \vec{P} \bullet \nabla \mathbf{Q} \\ \text{or} \\ U_{qp} = \mathbf{Q} : \nabla\nabla\Phi_p \end{array} \right) .$$

show that $U_{pq} = U_{qp}$.

3. Use a multipole expansion to find the potential field of three equal mass points located at the vertices of an equilateral triangle with side d. Restrict your solution to the plane of the triangle and keep only the first two terms of the expansion.

4. Find the interaction energy of a 10 kg sphere with the Earth-Moon system when the three are located so as to form an equilateral triangle. Assume the Earth and Moon are spherical. Compare the relative importance of the first two terms of the multipole expansion for the Earth-Moon potential.