6

The Two Body Problem

The classical problem of celestial mechanics, perhaps of all Newtonian mechanics, involves the motion of one body about another under the influence of their mutual gravitation. In its simplest form, this problem is little more than the generalization of the central force problem, but in some cases the bodies are of finite size and are not spherical. This may complicate the problem immensely as the potential fields of the objects no longer vary as the inverse square of the distance. This causes orbits to precess and the objects themselves to undergo gyrational motion. This latter motion results from external torques produced on a non-spherical object interacting with the object's own spin angular momentum. While we will not deal with the more difficult aspects of these phenomena in this book, it is useful to understand something of the properties of finite rigid bodies so that we are equipped to begin to understand some of the difficulties when they arise. Thus, we will begin our discussion of the two-body problem with a summary of the properties of rigid bodies.

6.1 The Basic Properties of Rigid Bodies

Let us begin by assuming that the rigid object we are considering is located in some orthonormal coordinate system so that the points within the object can be located in terms of some vector $\vec{r}$.

a. The Center of Mass and the Center of Gravity

Let us define two concepts usually taken for granted in mechanics books. First the center of mass is simply a 'mass weighted' mean position for the object. Again I will give both the discrete and continuous forms so that

$$r_c = \sum_i m_i \vec{r}_i \Big/ \sum_i m_i = \int_V \vec{r}\rho(\vec{r})dV \Big/ M \quad . \tag{6.1.1}$$

A second concept that is often confused with the center of mass is the *center of gravity*. This is often defined to be that point where the force of gravity can be considered to be acting. Mathematically that would mean that all torques produced by gravity would vanish about that point so that

$$\vec{r}_g \times \sum_i \vec{f}_i = \sum_i \vec{r}_i \times \vec{f}_i = \int_V \left[\vec{r} \times \rho(\vec{r})\vec{g}\right] dV = 0 \quad . \tag{6.1.2}$$

In a Cartesian coordinate frame this could be expressed in coordinate form as

$$\left.\begin{array}{l} r_{2,g}A_3 - r_{3,g}A_2 = B_{2,3} - B_{3,2} \\ r_{3,g}A_1 - r_{1,g}A_3 = B_{3,1} - B_{1,3} \\ r_{1,g}A_2 - r_{2,g}A_1 = B_{1,2} - B_{2,1} \end{array}\right\} \quad , \tag{6.1.3}$$

where

$$\left.\begin{array}{l} A_j = \sum_i g_{ij} m_i \\ B_{kj} = \sum_i r_{ik} g_{ij} m_i \end{array}\right\} \quad . \tag{6.1.4}$$

If one writes this as a linear system of equations for the components of the vector defining the center of gravity one gets

$$\mathbf{A}\vec{r} = \begin{pmatrix} 0 & A_3 & -A_2 \\ A_3 & 0 & A_1 \\ -A_2 & A_1 & 0 \end{pmatrix} \begin{pmatrix} r_{1,g} \\ r_{2,g} \\ r_{3,g} \end{pmatrix} = \begin{pmatrix} B_{2,3} - B_{3,2} \\ B_{3,1} - B_{1,3} \\ B_{1,2} - B_{2,1} \end{pmatrix} \quad . \tag{6.1.5}$$

However,

$$\text{Det}\ \mathbf{A} = 0 \quad . \tag{6.1.6}$$

This means that the equations are singular and there is no unique definition, so that the magnitude of r_g is undefined. Only if we require that $|\vec{r}_g| = |\vec{r}_c|$ and that the gravity vector be constant can we define a unique vector which will be equal to the vector to the center of mass. Thus, if the gravity field varies over the object, the center of gravity is not uniquely defined. In the case in which it is well defined it is the same as the center of mass. Physically one can see this by imagining all the points within a body where one could attach a hook suspend the object and not have it move. Any such points would serve as the center of gravity. The problem arises from the cross product and the definition. If one adds to the standard definition that the center of gravity is that point about which all the gravitational torques vanish *regardless of the orientation of the body with respect to the gravitational field*, then the definition is more tractable.

b. The Angular Momentum and Kinetic Energy about the Center of Mass

Consider that the object is rotating about some point that is fixed with respect to an inertial coordinate frame (i.e. one that has no accelerative motions). Then the angular momentum of the object will be

$$\vec{L} = \sum_i m_i(\vec{r}_i \times \vec{v}_i) = \int_V \rho(\vec{r})(\vec{r} \times \vec{v})dV \quad , \qquad (6.1.7)$$

where

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i \quad . \qquad (6.1.8)$$

Since we are considering the object to be rigid, then all points within the body will rotate with the same *angular velocity* ω. If that were not true some points within the body would catch up with others while moving away from still others and we would not call the body rigid. This allows us to separate the rotational motion from the positions of points within the object. Thus by making use of the vector identities from Chapter 1 we may write the angular momentum of the object as

$$\vec{L} = \sum_i m_i[\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \sum_i [\vec{\omega} r_i^2 - \vec{r}_i(\vec{r}_i \bullet \vec{\omega})] \quad . \qquad (6.1.9)$$

Writing out equation (6.1.9) for each component of $\vec{L}$ we see that equation (6.1.9) can be re-written as

$$\vec{L} = \mathbf{I} \bullet \vec{\omega} \quad , \qquad (6.1.10)$$

where **I** is known as the moment of inertia tensor and has components

$$I_{jk} = \left\{ \begin{array}{ll} \sum_i m_i (r_i^2 - x_k^2) & \text{for } j = k \\ \sum_i m_i x_j x_k & \text{for } j \neq k \end{array} \right\} \quad . \tag{6.1.11}$$

Now the kinetic energy of a rotating object about some fixed point is just

$$T = \tfrac{1}{2}\sum_i m_i v_i^2 = \tfrac{1}{2}\int_V \rho(\vec{r}) v^2(\vec{r}) dV = \tfrac{1}{2}\int_V \rho(\vec{r})\, \vec{v} \bullet (\vec{\omega} \times \vec{r}) dV \quad . \tag{6.1.12}$$

Making use of the so-called vector triple product

$$\vec{A} \bullet (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \bullet \vec{C} = \vec{C} \bullet (\vec{A} \times \vec{B}) , \tag{6.1.13}$$

we can write this as

$$T = \tfrac{1}{2} \vec{\omega} \bullet \int_V \rho(\vec{r})(\vec{r} \times \vec{v}) dV = \tfrac{1}{2} \vec{\omega} \bullet \vec{L} \quad . \tag{6.1.14}$$

This can be expressed in terms of the moment of inertia tensor by replacing the angular momentum with equation (6.1.10) so that

$$T = \tfrac{1}{2} \vec{\omega} \bullet \mathbf{I} \bullet \vec{\omega} = \tfrac{1}{2} \omega^2 (\hat{n} \bullet \mathbf{I} \bullet \hat{n}) = \tfrac{1}{2} \omega^2 I \quad . \tag{6.1.15}$$

here $\hat{n}$ is a unit vector pointing in the direction of the angular velocity vector and the quantity in square brackets is then just a property of the body and is called the moment of inertia about the axis $\hat{n}$. Clearly the moment of inertia tensor, **I** , will have the symmetric property

$$I_{ij} = I_{ji} \quad . \tag{6.1.16}$$

c. The Principal Axis Transformation

Calculations involving the moment of inertia tensor would be a lot easier if there were some coordinate frame in which the tensor were diagonal. It is clear from equation (6.1.11) that the tensor is a symmetric tensor so that the off diagonal terms satisfy

$$I = (\hat{n} \bullet \mathbf{I} \bullet \hat{n}) = \int_V \rho(\vec{r})(r^2 - \vec{r} \bullet \hat{n}) dV \quad . \tag{6.1.17}$$

Thus in order to make the tensor diagonal we need only transform to a coordinate frame wherein the off-diagonal elements are zero. We saw in Chapter 2 that one could reach any orthonormal coordinate frame from any other through a series of three coordinate rotations about the successive coordinate axes. This is represented by three independent parameters in the transformation (i.e. the rotation angles). Since we have three constraints to meet (i.e. making the off-diagonal elements zero), it is clear that this can be done. Another way of visualizing this transformation is to scale the unit vector $\hat{n}$ by $\sqrt{I}$ so that

$$\vec{\xi} = \sqrt{I}\,\hat{n} \quad . \tag{6.1.18}$$

In terms of the components of this vector the expression for the moment of inertia given by equation (6.1.17) becomes

$$I_{11}\xi_1^2 + I_{22}\xi_2^2 + I_{33}\xi_3^2 + I_{12}\xi_1\xi_2 + I_{13}\xi_1\xi_3 + I_{23}\xi_2\xi_3 = 1 \quad , \tag{6.1.19}$$

which is the general equation for an ellipsoid. Now there always is a coordinate frame aligned with the principal axes of the ellipsoid where the general equation for the surface becomes

$$I'_1(\xi'_1)^2 + I'_2(\xi'_2)^2 + I'_3(\xi'_3)^2 = 1 \quad . \tag{6.1.20}$$

This coordinate system is known as the *principal axis coordinate system* and it is the coordinate frame in which the off-diagonal elements of the moment of inertia tensor vanish. The diagonal elements are known as the principal moments of inertia, as they are indeed the moments of inertia about the principal axes. They are basically the eigenvalues of the moment of inertia tensor and so can be found from the determinental equation

$$\mathrm{Det}\begin{vmatrix} (I_{11}-I) & I_{12} & I_{13} \\ I_{21} & (I_{22}-I) & I_{23} \\ I_{31} & I_{32} & (I_{33}-I) \end{vmatrix} = 0 \quad , \tag{6.1.21}$$

which is nothing more that a polynomial in I. The principal moments of inertia are the roots of that polynomial.

The moment of inertia is an important concept if one is interested in the motion of an object. For example, it is essential for the understanding of precession. In the rotational equations of motion for an object the moment of inertia plays the role taken by the mass in the dynamical equations of motion of a system of particles.

6.2 The Solution of the Classical Two Body Problem

In principle we have assembled all the tools and concepts needed to solve some very difficult mechanics problems. To illustrate the methods needed to determine planetary motion we will consider the classical two body problem of celestial mechanics. We know immediately that we will have two second order vector differential equations to solve for the motion of both objects. Each of these equations will require six independent constants to specify the complete solution. Therefore we may expect to have to find a total of twelve constants of the motion before we can consider the problem solved.

a. The Equations of Motion

In order to find the equations of motion for two bodies moving under their mutual gravity we shall follow much the same procedure that we did for a central force. In order to keep the problem simple we will further assume that the potential of each body is that of a point mass m_l and m_2 respectively. The kinetic and potential energies of the system are then

$$\left.\begin{aligned} T &= \tfrac{1}{2}m_1(\dot{\vec{r}}_1 \bullet \dot{\vec{r}}_1) + \tfrac{1}{2}m_2(\dot{\vec{r}}_2 \bullet \dot{\vec{r}}_2) \\ V &= Gm_1m_2 \big/ |\vec{r}_1 - \vec{r}_2| \end{aligned}\right\} \quad . \tag{6.2.1}$$

where $\vec{r}_1$ and $\vec{r}_2$ are position vectors to the objects. These vectors are linearly independent so they form a suitable set of generalized coordinates in which to formulate the Lagrangian equations of motion. Now the elements that enter into the Lagrangian equations of motion are

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} &= m_i \dot{\vec{r}}_i \\ \frac{\partial \mathcal{L}}{\partial \vec{r}_i} &= \frac{\partial V}{\partial r_i} = -\frac{Gm_1m_2(\vec{r}_i - \vec{r}_j)}{d_{ij}^3} \end{aligned}\right\} \quad . \tag{6.2.2}$$

where

$$d_{ij} \equiv |\vec{r}_i - \vec{r}_j| \quad . \tag{6.2.3}$$

This leads to two vector equations of motion for the two bodies:

$$\left.\begin{aligned} m_1\ddot{\vec{r}}_1 + Gm_1m_2(\vec{r}_1 - \vec{r}_2)\big/d_{12}^3 &= 0 \\ m_2\ddot{\vec{r}}_2 + Gm_1m_2(\vec{r}_2 - \vec{r}_1)\big/d_{12}^3 &= 0 \end{aligned}\right\} \quad . \tag{6.2.4}$$

If we add these equations we get

$$m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = 0 \quad , \tag{6.2.5}$$

which can be integrated immediately twice with respect to time to yield

$$m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{A}t + \vec{B} \quad . \tag{6.2.6}$$

Note that $\vec{A}$ and $\vec{B}$ are vectors and so contain six linearly independent constants. From the definition of the center of mass [equation (6.1.1)] we can write

$$\mathcal{M}\vec{r}_c = \vec{A}t + \vec{B} \quad , \tag{6.2.7}$$

which says that at time t = 0 the center of mass was located at $(\vec{B}/\mathcal{M})$ and was moving with a uniform velocity $(\vec{A}/\mathcal{M})$. Thus we have immediately found six of the twelve constants of the motion. They are the location and velocity of the center of mass.

Since a coordinate frame that undergoes uniform motion is an inertial coordinate frame (i.e. no accelerations) the laws of physics will look the same in a coordinate frame moving with the center of mass as they did in our initial coordinate system. Therefore we will transform to an inertial coordinate frame with the origin located at the center of mass. In such a coordinate system

$$m_1\vec{r}'_1 + m_2\vec{r}'_2 = 0 \quad . \tag{6.2.8}$$

We may use this constraint to decouple each of equations (6.2.4) from the other so that

$$\left.\begin{aligned} \ddot{\vec{r}}'_1 + \frac{G(m_1+m_2)\vec{r}'_1}{d_{12}^3} = 0 \\ \ddot{\vec{r}}'_2 + \frac{G(m_1+m_2)\vec{r}'_2}{d_{12}^3} = 0 \end{aligned}\right\} \quad . \tag{6.2.9}$$

We can reduce these further by introducing a new vector that runs from one object to the other so that

$$\vec{r} = \vec{r}'_1 - \vec{r}'_2 \quad . \tag{6.2.10}$$

Then by subtracting the second of equations (6.2.9) from the first we get

$$\ddot{\vec{r}} + \frac{GM\vec{r}}{d_{12}^3} = 0 \quad . \tag{6.2.11}$$

This is equivalent to making another coordinate transformation to one of the objects since $\vec{r}$ is simply the distance between the objects. However, this reduces the problem to the one we solved in the previous chapter, since the form of equation (6.2.11) is the same as equation (5.1.3). Thus the solution of the two body problem is equivalent to the solution of a central force problem where the potential is the gravitational potential and the source of the force can be viewed as being located in one of the objects.

Thus we may jump directly to the solution of the problem given by equations (5.4.9 -5.4.12) and write

$$\left.\begin{aligned} r &= \frac{P}{[1+e\cos(\theta-\theta_0)} \\ P &= \frac{L^2}{G\mathcal{M}m^2} \\ e &= \left[\frac{1+2EL^2}{(G\mathcal{M}m)^2 m}\right]^{1/2} \end{aligned}\right\} \quad . \tag{6.2.12}$$

Here we have found three more constants in E, L, and θ_0. We knew that the angular momentum and the energy would have to be two of the constants, and that an initial value of θ_0 is involved should be no surprise. While equations (6.2.12) introduce the angular momentum, they only specify its magnitude, and we know from the central force problem that the *vector* is an integral of the motion. That is what insures that the motion is planar. Therefore specifying the angular momentum specifies two additional linearly independent components (in addition to the magnitude). The last remaining constant is the r_o that appears in equation (5.3.3) and specifies the location of the particle in its orbit at some specific time. Like θ_0, it can be regarded as an initial value of the problem. Thus we have all six remaining constants of the motion containing sufficient information to uniquely determine the position of each object in space as a function of time.

b. Location of the Two Bodies in Space and Time

By choosing a coordinate system with its origin at one of the bodies, we are really only concerned with describing the motion of one of the objects with respect to the other. While equations (6.2.12) indicate the shape of the orbit, they say nothing about how the object moves in time. To describe the motion, we shall have to make use of Kepler's second law, the constancy of the areal velocity. To do this we shall have to introduce some new terminology.

As an example, let us consider the motion of an object about the sun. Since we want to describe the motion of an object in its orbit, we shall need some means to define specific locations in the orbit as reference points and parameters to measure angular positions. We shall presume that the orbit is elliptical with the sun at one focus in accord with Kepler's first law, Thus there will be a point in the orbit where the object makes it closest approach to the sun, This point is known as *perihelion* since, in general, the point of closest approach to the source of the force-field is known as *peri**** , where *** is the Greek stem appropriate to the object. This point is always located at one end of the semi- major axis of the ellipse. In the case of orbits about the sun, the other end of the semi-major axis is known as *aphelion* and is the position furthest from the sun. Since the origin of the coordinate system is at the source of the attractive force, the location of the object in its orbit can be defined by an angle measured from the semi-major axis - specifically from the point of perihelion (see Figure 6.1) in the direction of the object's motion. This angle is called the true anomaly, and will be denoted by the Greek letter ν. Determining it as a function of time essentially solves the problem of finding the temporal location of the object.

Let us choose to start measuring time from perihelion passage so that the true anomaly is zero when t = 0. From the solution to the orbit equation [equation (6.2.12)] we see that t = 0 will occur when $\theta = \theta_0$ so that'

$$\nu = \theta - \theta_0 \quad . \tag{6.2.13}$$

We may then write the orbit solution as

$$r = \frac{P}{1 + e\cos\nu} = \frac{a(1-e^2)}{1 + e\cos\nu} \quad , \tag{6.2.14}$$

where a is the semi-major axis of the ellipse.

Now we shall appear to digress to some geometry and relate each point on the elliptical orbit to a corresponding point on a circle with a radius equal to the semi-major axis and whose center is located at the center of the ellipse (again see Figure 6.1). An ellipse is simply the projection of a circle that has been rotated about its diameter through some angle ψ. Now imagine points $[x_c, y_c]$ located on the circle and corresponding points $[x_e, y_e]$ located on the ellipse, For $x_c = x_e$,

$$\frac{y_e}{y_c} = \frac{b}{a} = \cos\psi \quad , \tag{6.2.15}$$

where a and b are the semi-major and semi-minor axes of the ellipse respectively. Since cosψ is the same for all corresponding $(x_c = x_e)$ points on the circle and the ellipse, this result must hold for all such points. The Pythagorean Theorem assures us that

$$r^2 = y_e^2 + (f - x_e)^2 = (b/a)^2 y_c^2 + (f - x_c)^2 \quad , \tag{6.2.16}$$

where f is the distance from the center to the focus of the ellipse. From the equation for the ellipse [see equation (6.2.14)], we can write for ν = 0 that

$$r = a - f = a(1-e^2)/(1+e) = a(1-e) \quad , \tag{6.2.17}$$

which becomes

$$f = ae = (a^2 - b^2)^{1/2} \quad . \tag{6.2.18}$$

If we define an angle E measured from perihelion to a point on the circle $[x_c, y_c]$ as seen from the center of the circle, then

$$\left.\begin{array}{l} x_c = a\cos(E) \\ y_c = a\ \sin(E) \end{array}\right\} \quad . \tag{6.2.19}$$

Using these definitions, $b^2 = a^2(1-e^2)$, and equation (6.2.18), equation (6.2.16) becomes

$$r = a[1 - e\cos(E)] \quad . \tag{6.2.20}$$

The angle (E) is called the *eccentric anomaly*. Now we are in a position to relate the areal velocity of the particle along the elliptic orbit to the areal velocity of an imaginary particle along the circle.

Imagine such a particle moving in a circle with a radius equal to the semi-major axis (a) of the ellipse. Both particles would have the same orbital period since that depends only on the semi-major axis. However, the imaginary particle moving on the circle would move along its orbit at a uniform rate of speed. Therefore let us define its angular rate of speed as

$$n = \frac{M}{t} = \frac{2\pi}{P} \quad , \tag{6.2.21}$$

where P is the orbital period. Here M is the angular distance along the circle that the imaginary particle would have moved during the time t specifying the position of the real particle on the ellipse. Thus

$$M = nt \quad . \tag{6.2.22}$$

The angle M is called the *mean anomaly*.

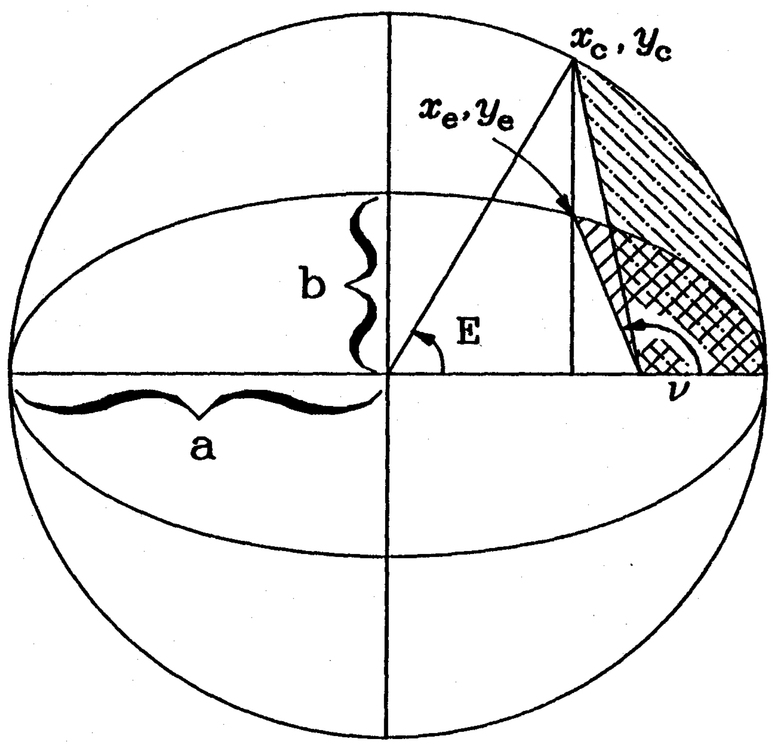

Figure 6.1 shows the geometrical relationships between the elliptic orbit and the osculating circle. The areas swept out by radius vectors to points on the ellipse and the circle are shown as the shaded areas. By relating the sides of the bounded figures, we may relate the area swept out in the ellipse to the area swept out on the circle of a uniformly moving object. This is the source of Kepler's equation.

We may relate the mean anomaly to the eccentric anomaly by the following argument. From the law of areas (Kepler's second law)

$$\frac{M}{2\pi} = \frac{A}{\pi ab} \quad , \tag{6.2.23}$$

where A is the area swept out by the radius vector in time t while πab is just the area of the ellipse. Now, since each point on the circle is simply a scaled point on the ellipse, the areas in equation (6.2.23) scale by (a/b) so that

$$\frac{M}{2\pi} = \frac{B}{\pi a^2} = \frac{\frac{1}{2}a^2 E - \frac{1}{2} f y_c}{\pi a^2} = \frac{\frac{1}{2}a^2 E - \frac{1}{2}a^2 e\sin(E)}{\pi a^2} \quad , \tag{6.2.24}$$

where B is the dot-dashed area of Figure 6.1 so that

$$M = E - e\sin(E) \quad . \tag{6.2.25}$$

This expression is known as Kepler's equation since it specifically utilizes Kepler's second law to relate the mean anomaly to the eccentric anomaly. We may use equation (6.2.20) and the equation for an ellipse [equation (6.2.14)] to relate the eccentric anomaly to the true anomaly. By equating the value of r given by each of these equations, we get

$$\frac{a(1-e^2)}{1+e\cos\nu} = a[1-e\cos(E)] \quad , \tag{6.2.26}$$

which after some trigonometry becomes:

$$\tan(\nu/2) = \left[\frac{1+e}{1-e}\right]^{1/2} \tan(E/2) \quad . \tag{6.2.27}$$

Equation (6.2.27) and Kepler's equation [equation (6.2.25)] , therefore, relate the time since perihelion passage to the true anomaly or angular position of the real object in its elliptic orbit. The conservation of angular momentum leads to similar results for hyperbolic and parabolic orbits. Specifically for hyperbolic orbits we have

$$\left.\begin{aligned} r &= a[e\cosh(F) - 1] \\ M &= e\sinh(F) - F \\ \tan(\nu/2) &= [(e+1)/(e-1)]^{1/2} \tanh(F/2) \end{aligned}\right\} \quad , \tag{6.2.28}$$

while for parabolic orbits we get

$$\left.\begin{aligned} &r = q\sec^2(\nu/2) = q[1+\tan^2(\nu/2)] \\ &2x = 3M + [9M^2+4]^{1/2} \\ &\tan(\nu/2) = x^{(1/3)} - x^{-(1/3)} \end{aligned}\right\} \quad . \qquad (6.2.29)$$

The quantity n, which is the mean daily motion, has the same physical interpretation for both the elliptic and hyperbolic orbits, but it is defined slightly differently for parabolic orbits.

From Newton's laws of motion and gravitation we can write the mean daily motion for objects in elliptic orbit as

$$n = 2\pi/P = (G\mathcal{M}/a^3)^{1/2} \quad , \qquad (6.2.30)$$

where *M* is the *sum* of the masses of the two bodies. However, in the solar system we can use the earth's orbital parameters as units to define the motion of objects about the sun and express n in those units and a constant k, known as the Gaussian constant as

$$n = k[(\mathcal{M}/\mathcal{M}_{u})/(a/a_{\oplus})^3]^{1/2} \quad \text{radians/day} \quad . \qquad (6.2.31)$$

Actually the value of k is taken to be

$$k=0.01720209895 \quad \text{radians/day} \quad , \qquad (6.2.32)$$

and its value is used to define the astronomical unit. Generally one hears that the astronomical unit is the semi-major axis of the earth's orbit by definition, but this is not strictly correct. It is k that is fixed with units of mass measured in solar masses, time in ephemeris days, and the unit of length is the astronomical unit by definition. Indeed, using the modern value for the mass of the earth (in units of the solar mass) one would find that the semi-major axis of the earth's orbit is about $(1+3\times10^{-7})$ astronomical units. Brouwer and Clemence[5] point out that Kepler's third law isn't strictly correct if there is a massive third body in the system so the fact that the semi-major axis of the earth's orbit is not exactly one astronomical unit should not be a bother. As long as the unit of length is well defined by equation (6.2.31), we may use it to determine the mean angular motion for objects in the solar system.

The analogous expressions for hyperbolic and parabolic orbits are

$$\left.\begin{array}{ll} n = k[(M/M_u)/(a_h/a_\oplus)^3]^{1/2} & \text{Hyperbolic orbits} \\ n = k[(M/M_u)/2(q/a_\oplus)^3]^{1/2} & \text{Parabolic orbits} \end{array}\right\} \quad . \tag{6.2.33}$$

Here a_h is called the semi-transverse axis of the hyperbola and q is known as the pericentric distance which is simply the distance of closest approach to the second object. In the solar system the sun's mass so dominates that M/M_u is effectively unity. Thus if we know the type of orbit and orbital scale-length (i.e. semi-major axis for the ellipse, semi-transverse axis for the hyperbola, or pericentric distance for the parabola) we can determine the mean daily motion from equations (6.2.31 - 6.2.33). Further knowledge of the time since perihelion passage allows the calculation of the mean anomaly M. That and the eccentricity enable us to calculate the eccentric anomaly through the solution of Kepler's equation. Algebra, in the form of equations (6.2.27-6.2.29), allows for the calculation of the true anomaly and the radial distance r from the origin of the coordinate system. This, then completely specifies the location of the object in its orbit. Involved as this process is, it is relatively straightforward except for the solution of Kepler's equation.

c. The Solution of Kepler's Equation

Equations of the form of equation (6.2.25) are known as transcendental equations and, in general do not have closed form solutions. Thus, in order to solve the problem of orbital motion, we will be forced to a numerical solution of Kepler's equation. Much has been written on effective and general numerical procedures for such a solution and we will not go into all of those details here. Rather we shall adapt a common numerical procedure known as Newton-Raphson iteration. Assume that we have an equation of the form

$$f(x) = 0 \quad , \tag{6.2.34}$$

and we wish to find that value of x for which the equation is satisfied. A procedure for accomplishing this is to guess an initial value $x^{(0)}$ and use the following expression to improve it.

$$x^{(k+1)} = x^{(k)} - \frac{f[x^{(k)}]}{f'[x^{(k)}]} \quad . \tag{6.2.35}$$

The process is repeated until

$$\left|\frac{[x^{(k+1)} - x^{(k)}]}{x^{(k)}}\right| \le \varepsilon \quad , \qquad (6.2.36)$$

where ε is some predetermined tolerance. The value of x for which ε = 0 is known as the 'fixed-point' of the iteration scheme and a rather large body of knowledge has been developed concerning such schemes. The specific one given in equation (6.2.35) has the virtue of normally converging quickly to a fixed point and is simple. It is called Newton-Raphson iteration and graphically amounts to extending a tangent to the function $f[x^{(k+1)}]$ to the point where it intercepts the x-axis and using that value of x as $x^{(k+l)}$ .Clearly, when f(x) is zero, x is a fixed point. The application of the method to Kepler's equation yields

$$\left.\begin{aligned} E^{(0)} &= M + e\sin(M) \\ E^{(k+1)} &= E^{(k)} + \frac{M - E^{(k)} + e\sin[E^{(k)}]}{1 - e\cos[E^{(k)}]} \end{aligned}\right\} \quad . \qquad (6.2.37)$$

One of the problems with the Newton-Raphson scheme is that it doesn't always converge. This is the case with equations (6.2.37). There are values of the eccentricity and mean anomaly for which this iteration scheme will not yield an answer. However, this occurs only for a small range of M near perihelion and very large eccentricities (see Chapter 6 exercises). It will always work for objects in elliptical orbits in the solar system except for some long period comets and these orbits may be handled in another manner. Thus, for simplicity, we will leave the discussion of the solution of Kepler's equation with the Newton-Raphson iteration scheme. Those who wish more details on the subject should consult Green[6].

6.3 The Orientation of the Orbit and the Orbital Elements

The solution to the two body problem consists in describing the motion of both bodies in an arbitrary coordinate frame. Since the two bodies are described by two vector differential equations of second order, there will be twelve constants required for that description. Six of those twelve are required to describe the motion of the center of mass of the system. Three more are required to locate one object in its orbit relative to the other. The remaining three are required to specify the orientation of the orbit with respect to the arbitrary coordinate frame. If we assume that the coordinate frame is a spherical coordinate frame, then we can use the Euler angles as defined in Chapter 2 to define the orbital orientation in that frame. The coordinate frame will have a fundamental plane and a direction within that plane that defines how azimuthal angles will be measured. For most astronomical coordinate systems of relevance to celestial mechanics, that direction is toward the first point of Aries (i.e. the vernal equinox) and the fundamental plane will be either the ecliptic or the equator of the earth (see Chapter 2).

Figure 6.2 shows the orbit of an object located in the reference coordinate frame and it bears a marked similarity to the last of Figures 2.2. In Figure 2.2 ϕ described the distance from the preferred direction to the line of intersection of the two planes known as the line of nodes. In celestial mechanics, this is known as the longitude of the ascending node where the notion of "ascending" refers to that node where the motion of the object carries it toward positive Z. In the solar system, this means that the object would be moving from south to north in the sky. We will use Ω to denote this angle. The second of the Euler angles in Figure 2.2 is θ and measures the angle by which one plane is inclined to the other. In celestial mechanics this is known as the angle of inclination and is usually denoted by i. The last of the Euler angles in Figure 2.2 is ψ and is used to denote a particular point in the inclined plane. For orbital mechanics the most logical point in the orbit is the pericenter. Its location is then designated by the angle $\mathbf{o}$ called the argument of the pericenter. Thus the three defining angles of the orbit are

$$\left.\begin{aligned} &\Omega \equiv \textit{The Longitude of the Ascending Node} \\ &i \equiv \textit{The Inclination of the Orbit } (\textit{measured from } v = 0^{o} \rightarrow 180^{o}) \\ &\mathbf{o} \equiv \textit{The Argument of the Pericenter} \\ &\quad \textit{(measured from the ascending node in the direction of motion} \\ &\quad \textit{with a range } 0^{o} \rightarrow 180^{0}\textit{)} \end{aligned}\right\} \quad . \tag{6.3.1}$$

Sometimes the argument of the pericenter is replaced by the strange angular sum ($\mathbf{o} + \Omega$) which is called the longitude of the pericenter and is denoted by

$$\varpi = \Omega + \mathbf{o} \equiv \text{The Longitude of the Pericenter} \quad . \tag{6.3.2}$$

Thus we have defined the three remaining constants required by the equations of motion specifying the orientation of the orbital plane. In the solar system, the center of attraction is usually the sun and so the pericenter becomes perihelion and the fundamental plane is usually the ecliptic.

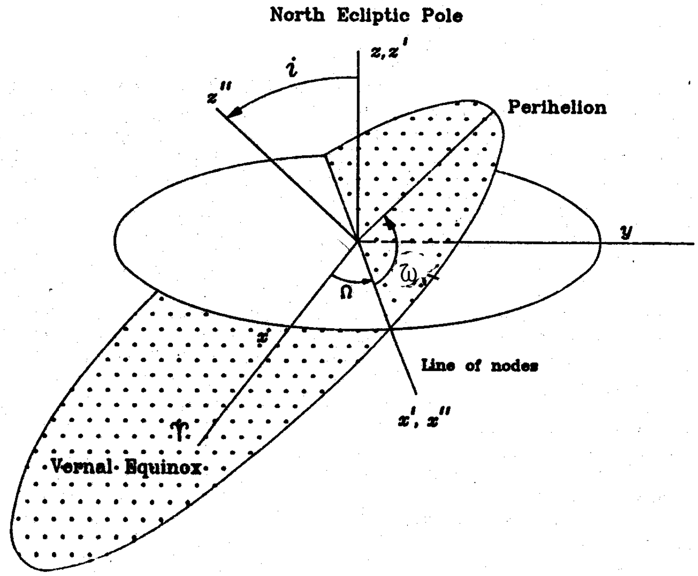

Figure 6.2 shows the coordinate frames that serve to define the orbital elements specifying the orientation of the orbit with respect to the ecliptic coordinate system.

We have repeatedly said that there are twelve constants required to uniquely specify the motion of one object about another, but that six of them are concerned with the motion of the center of mass of the pair. Since this motion is uniform, these six constants are usually ignored when discussing the orbit of the object. The remaining six constants constitute the elements of the orbit and can be broken into two sets of three. The three that define the orientation of the orbit as defined above are taken directly as orbital elements. However, the remaining three that specify the size and shape of the orbit as well as the object's location in it at some time can be specified in various ways. We found in Chapter 4 that the angular momentum and total energy are integrals of the motion and will determine the size and shape of the orbit. However, they are not directly observable quantities so that a different set of constants more directly related to the geometry of the orbit is usually chosen to represent the orbit. These are the semi-major axis and the eccentricity. Finally to represent the position of the object within its orbit we specify the time when the object is at pericenter, or for the solar system, the time of perihelion passage T_0. Now in developing the equations describing the motion of the object in its orbit, we took the time of perihelion passage to be zero. Thus (t) in equation (6.2.21) and equation (6.2.22) should be replaced by

$$t = t - T_0 \quad . \tag{6.3.3}$$

The six constants specifying the motion of the object are known as the elements of the orbit of the object and are:

$$\begin{aligned}
&a \equiv \text{The Semi-major axis of the orbit} \\
&e \equiv \text{The Orbital Eccentricity} \\
&T_0 \equiv \text{The Time of Perihelion Passage} \\
&\mathbf{o} \equiv \text{The Argument of Perihelion} \\
&\Omega \equiv \text{The Longitude of the Ascending Node} \\
&i \equiv \text{The Inclination of the Orbit} \quad .
\end{aligned} \tag{6.3.4}$$

While we have now located the object in its orbit, we have yet to find it in the sky.

6.4 The Location of the object in the Sky

The location of the object in the sky involves nothing more than the transformation from the coordinate system specifying the location of the object in its orbit to the coordinate system of the observer. The specific nature of this transformation depends on the relative location of the source of the attractive force and observer. For example, we will consider the object to be in orbit about the sun and the observer located on a spinning earth. Since the heliocentric orbital elements are generally referred to the ecliptic, the first part of the transformation will involve expressing the components of the radius vector to the object in ecliptic coordinates. Then we will transform to the equatorial (Right Ascension-Declination) coordinate system. This is followed by shifting the origin of the coordinate system to the center of the earth and finally the astronomical triangle may be solved to express the result in the observer's Alt-Azimuth coordinate system.

Imagine a Cartesian coordinate system with its origin coinciding with the sun, the z-axis normal to the orbit plane, and the x-axis passing through perihelion. In such a coordinate system the components of the radius vector to the orbiting object are

$$\vec{r} = \begin{bmatrix} r\cos\nu \\ r\sin\nu \end{bmatrix} \quad . \tag{6.4.1}$$

We wish to transform this coordinate frame to the equatorial coordinate frame. Therefore we first carry out the inverse Euler rotational transformations that will align the x-axis with the direction to the vernal equinox and the z-axis normal to the plane defining the orbital elements (usually the ecliptic plane). This will yield the components of the vector in ecliptic coordinates as

$$\vec{r}' = \mathbf{P}_z^{\mathbf{T}}(\Omega)\mathbf{P}_x^{\mathbf{T}}(i)\mathbf{P}_z^{\mathbf{T}}(\mathbf{o})\vec{r} \quad . \tag{6.4.2}$$

Now to express the coordinates in Right Ascension-Declination coordinates, we must align the defining planes of the two coordinate systems. This can be accomplished by a rotation about the x-axis, pointing toward the vernal equinox, through an angle $-\varepsilon$ where ε is the angle between the ecliptic and equatorial planes. Note that a rotation through a negative angle is equivalent to the inverse transformation of the positive rotation. Thus the radius vector can be expressed in heliocentric equatorial coordinates as

$$\vec{r}'' = \mathbf{P}_x^{\mathbf{T}}(\varepsilon)\vec{r}' \quad . \tag{6.4.3}$$

Now the origin of the coordinate system must be transferred to the earth. This is a vector transformation and is accomplished by simply subtracting a heliocentric vector to the earth from the heliocentric vector locating the object. Thus a radius vector from the earth to the object will have geocentric equatorial coordinates of

$$\vec{\rho} = [\ \mathbf{P}_x^{\mathbf{T}}(\varepsilon)\,\mathbf{P}_z^{\mathbf{T}}(\Omega)\,\mathbf{P}_x^{\mathbf{T}}(i)\,\mathbf{P}_z^{\mathbf{T}}(\mathbf{o})\ ]\ \vec{r} - \vec{X}_\oplus \quad . \tag{6.4.4}$$

Here the vector $X_\oplus$ is the heliocentric equatorial radius vector to the earth.

Having arrived at the earth, we need only correct for the observer's location on the earth. Remember that the x-axis is still pointing at the vernal equinox and the z-axis toward the north celestial pole. Thus to get to the local alt-azimuth coordinate system, we must align the x-axis with the local prime meridian (pointing north) and then bring the z-axis so that it points toward the zenith. The first of these transformations can be accomplished by rotating about the z- axis (polar axis) through the local hour angle of the vernal equinox, but this is just the local sidereal time by definition. At this point the x-axis will lie in the plane of the prime meridian, but pointing south (in the northern hemisphere) so we must rotate through an additional angle of 180°. If the object happens to be close by, it may finally be necessary to transfer the origin from the center of the earth to the observer by subtracting the radius vector from the center of the earth to the observer's location. Following this by a rotation through the co-latitude of the observer will bring the z-axis so that it points toward the zenith. Thus the complete transformation from the orbital coordinates of equation (6.4.1) to the true topocentric coordinates of the observer can be written as

$$\vec{\rho}' = \mathbf{P}_y[(\pi/2)-\delta]\,\mathbf{P}_z[h(t)]\left\{[\ \mathbf{P}_x^{\mathbf{T}}(\varepsilon)\,\mathbf{P}_z^{\mathbf{T}}(\Omega)\,\mathbf{P}_x^{\mathbf{T}}(i)\,\mathbf{P}_z^{\mathbf{T}}(\boldsymbol{\omega})\ ]\ \vec{r} - \vec{X}_\oplus\right\} - \vec{r}_\oplus\ . \quad (6.4.5)$$

If the transformation from the center of the earth to the true topocentric coordinates is carried out as indicated by equation (6.4.5), the vector $\vec{r}_\oplus$ has only an x-component equal to the radius of the earth for the observer's latitude and longitude. The components of the vector from the observer have the following components in the Alt-Azimuth coordinate system:

$$\vec{\rho} = \begin{pmatrix} \rho\sin(H) \\ \rho\cos(H)\cos(A) \\ \rho\cos(H)\sin(A) \end{pmatrix}\ , \quad (6.4.6)$$

which translates into the Alt-Azimuth coordinates of,

$$\left.\begin{array}{l} \rho^2 = \rho_x^2 + \rho_y^2 + \rho_z^2 \\ \tan A = \dfrac{\rho_y}{\rho_x} \\ \sin H = \dfrac{\rho_x}{\rho} \end{array}\right\}\ . \quad (6.4.7)$$

Thus we have completely described the motion of an object around the sun to the point where we can locate the object in the sky. In the next chapter we shall consider the inverse problem of determining the orbital elements from observation.

Chapter 6: Exercises

1. Given a body which is bounded by the surface

$$x^2(b^2+3a^2)+2\sqrt{3}(xy)(b^2-a^2)+y^2(3b^2+a^2) + (4a^2b^2/c^2)z^2 = 4a^2b^2 \quad ,$$

where $a > b > c$, and has a density distribution $\rho(r) = \text{const.}$ Find the principal moments of inertia and the principal axes of the body.

2. Integrate the equations of motion for the two-body problem to show that

$$e = (1+2EL^2/mk^2)^{1/2} \quad .$$

3. Assuming the earth's orbit to be circular and that meteors approach the sun in parabolic orbits, between what limits on their relative speed will they hit the earth if the gravitational attraction of the earth is neglected?

4. Consider two particles orbiting about one another and having masses m_1 and m_2. If the force between the two is given by

$$\vec{F} = k^2(\vec{r}_1 - \vec{r}_2)\,,$$

show that the orbit of one particle about the other is an ellipse with one particle at the *center* of the ellipse.

5. A rocket is detected approaching Chicago at a range of 3200km, and an altitude of 160km above sea level. If the velocity of approach is 24800km/hr and the motion is parallel to the surface of the earth, decide if the rocket will hit Chicago. Assume that the earth is spherical and that coriolis forces and atmospheric drag are negligible. What are the values of r and ν at the instant of detection? If it should miss, how much will it miss by? If the azimuth at the time of detection is 15^o, where is the probable launch site?

6. Find the Right Ascension, declination, altitude and azimuth for Mars as seen from The Ohio State University campus on March 1, 1988 at 3:00AM EST. List all additional constants and their source necessary to solve this problem.

7. If one has an iterative function that can be written as

$$x^{(k+1)} = \mathsf{T}[f(x^{(k)})] \ ,$$

then it will converge to a fixed point if and only if

$$\left|\frac{\partial \Im}{\partial x}\right| < 1 \quad \forall\, x \quad \exists \left| x^{(k)} \right| < |\, x \,| < |\, x(\text{fixed} - \text{point}) \,| \quad .$$

Find the range of values of e and E for which Newton-Raphson iteration will converge to a solution of Kepler's equation.