9

Perturbation Theory and Celestial Mechanics

In this last chapter we shall sketch some aspects of perturbation theory and describe a few of its applications to celestial mechanics. Perturbation theory is a very broad subject with applications in many areas of the physical sciences. Indeed, it is almost more a philosophy than a theory. The basic principle is to find a solution to a problem that is similar to the one of interest and then to cast the solution to the target problem in terms of parameters related to the known solution. Usually these parameters are similar to those of the problem with the known solution and differ from them by a small amount. The small amount is known as a perturbation and hence the name perturbation theory.

This prescription is so general as to make a general discussion almost impossible. The word "perturbation" implies a small change. Thus, one usually makes a "small change" in some parameter of a known problem and allows it to propagate through to the answer. One makes use of all the mathematical properties of the problem to obtain equations that are solvable usually as a result of the relative smallness of the perturbation. For example, consider a situation in which the fundamental equations governing the problem of interest are linear. The linearity of the equations guarantees that any linear combination of solutions is also a solution. Thus, one finds an analytic solution close to the problem of interest and removes it from the defining equations. One now has a set of equations where the solution is composed of small quantities and their solution may be made simpler because of it.

However, the differential equations that describe the dynamics of a system of particles are definitely nonlinear and so one must be somewhat more clever in applying the concept of perturbation theory. In this regard, celestial mechanics is a poor field in which to learn perturbation theory. One would be better served learning from a linear theory like quantum mechanics. Nevertheless, celestial mechanics is where we are, so we will make the best of it. Let us begin with a general statement of the approach for a conservative perturbing force.

9.1 The Basic Approach to the Perturbed Two Body Problem

The first step in any application of perturbation theory is to identify the space in which the perturbations are to be carried out and what variables are to be perturbed. At first glance, one could say that the ultimate result is to predict the position and velocity of one object with respect to another. Thus, one is tempted to look directly for perturbations to $\vec{r}$ as a function of time. However, the non-linearity of the equations of motion will make such an approach unworkable. Instead, let us make use of what we know about the solution to the two body problem.

For the two body problem we saw that the equations of motion have the form

$$\ddot{\vec{r}} + \nabla\Phi = 0 \quad , \tag{9.1.1}$$

where Φ is the potential of a point mass given by

$$\Phi = -\frac{GM}{r} \quad . \tag{9.1.2}$$

Let us assume that there is an additional source of a potential that can be represented by a scalar -ψ that introduces small forces acting on the object so that

$$|\nabla\Phi| >> |\nabla\Psi| \quad . \tag{9.1.3}$$

We can then write the equations of motion as

$$\ddot{\vec{r}} + \nabla\Phi = \nabla\Psi(\vec{r}, t) \quad . \tag{9.1.4}$$

Here ψ is the negative of the perturbing potential by convention. If ψ is a constant, then the solution to the equations of motion will be the solution to the two body problem. However, we already know that this will be a conic section which can be represented by six constants called the orbital elements. We also know that these six orbital elements can be divided into two triplets, the first of which deals with the size and shape of the orbit, and the second of which deals with the orientation of that orbit with respect to a specified coordinate system. A very reasonable question to ask is how the presence of the perturbing potential affects the orbital elements. Clearly they will no longer be constants, but will vary in time. However, the knowledge of those constants as a function of time will allow us to predict the position and velocity of the object as well as its apparent location in the sky using the development in Chapters 5, 6, and 7. This results from the fact that at any instant in time the object can be viewed as following an orbit that is a conic section. Only the characteristics of that conic section will be changing in time. Thus, the solution space appropriate for the perturbation analysis becomes the space defined by the six linearly independent orbital elements. That we can indeed do this results from the fact that the uniform motion of the center of mass provides the remaining six constants of integration even in a system of N bodies. Thus the determination of the temporal behavior of the orbital elements provides the remaining six pieces of linearly independent information required to uniquely determine the object's motion. The choice of the orbital elements as the set of parameters to perturb allows us to use all of the development of the two body problem to complete the solution.

Thus, let us define a vector $\vec{\xi}$ whose components are the instantaneous elements of the orbit so that we may regard the solution to the problem as given by

$$\vec{r} = \vec{r}(\vec{\xi}, t) \quad . \tag{9.1.5}$$

The problem has now been changed to finding how the orbital elements change in time due to the presence of the perturbing potential $-\psi$. Explicitly we wish to recast the equations of motion as equations for $d\vec{\xi}/dt$. If we consider the case where the perturbing potential is zero, then $\vec{\xi}$ is constant so that we can write the unperturbed velocity as

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{\partial \vec{r}(\vec{\xi}, t)}{\partial t} \quad . \tag{9.1.6}$$

Now let us define a specific set of orbital elements $\vec{\xi}_0$ to be those that would determine the particle's motion if the perturbing potential suddenly became zero at some time t_0

$$\vec{\xi}_0 \equiv \vec{\xi}(t_0) \quad . \tag{9.1.7}$$

The orbital elements $\vec{\xi}_0$ represent an orbit that is tangent to the perturbed orbit at t_0 and is usually called the *osculating orbit*. By the chain rule

$$\frac{d\vec{r}}{dt} = \frac{\partial\vec{r}}{\partial t} + \frac{\partial\vec{r}}{\partial\vec{\xi}} \bullet \dot{\vec{\xi}} \quad . \tag{9.1.8}$$

Equation (9.1.6) is correct for ξ_0 and at $t = t_0$, $\xi = \xi_0$ so if we compare this with equation (9.1.6), we have

$$\frac{\partial\vec{r}}{\partial\vec{\xi}} \bullet \dot{\vec{\xi}} = 0 \quad . \tag{9.1.9}$$

This is sometimes called the osculation condition. Applying this condition and differentiating equation (9.1.8) again with respect to time we get

$$\frac{d^2\vec{r}}{dt^2} = \frac{\partial^2\vec{r}}{\partial t^2} + \left(\frac{\partial}{\partial\vec{\xi}}\frac{\partial\vec{r}}{\partial t}\right) \bullet \dot{\vec{\xi}} \quad . \tag{9.1.10}$$

If we replace d^2r/dt^2 in equation (9.1.4) by equation (9.1.10), we get

$$\frac{\partial^2\vec{r}}{\partial t^2} + \nabla\Phi + \frac{\partial^2\vec{r}}{\partial t^2} + \left(\frac{\partial}{\partial\vec{\xi}}\frac{\partial\vec{r}}{\partial t}\right) \bullet \dot{\vec{\xi}} = \nabla\Psi(\vec{r},t) \quad . \tag{9.1.11}$$

However the explicit time dependence of $\vec{r}(\vec{\xi},t)$ is the same as $\vec{r}(\vec{\xi}_0,t)$ so that

$$\frac{\partial^2\vec{r}}{\partial t^2} + \nabla\Phi = 0 \quad . \tag{9.1.12}$$

Therefore the first two terms of equation (9.1.11) sum to zero and, together with the osculating condition of equation (9.1.9), we have

$$\left.\begin{aligned} \dot{\vec{\xi}} \bullet \frac{\partial}{\partial\vec{\xi}}\left[\frac{\partial\vec{r}}{\partial t}\right] &= \nabla\Psi \\ \dot{\vec{\xi}} \bullet \frac{\partial\vec{r}}{\partial\vec{\xi}} &= 0 \end{aligned}\right\} \quad , \tag{9.1.13}$$

as equations of condition for E. Differentiation with respect to a vector simply means that the differentiation is carried out with respect to each component of the vector. Therefore $[\partial\vec{r}/\partial\vec{\xi}]$ is a second rank tensor with components $[\partial r_i/\partial\xi_j]$. Thus each of the equations (9.1.13) are vector equations, so there is a separate scalar equation for each component of $\vec{r}$. Together they represent six nonlinear inhomogeneous partial differential equations for the six components of $\vec{\xi}$. The initial conditions for the solution are simply the values of $\vec{\xi}_0$ and its time derivatives at t_0. Appropriate mathematical rigor can be applied to find the conditions under which this system of equations will have a unique solution and this will happen as long as the Jacobian of $\left|\partial(\vec{r},\vec{v})/\partial\vec{\xi} \neq 0\right|$. Complete as these equations are, their form and application are something less than clear, so let us turn to a more specific application.

9.2 The Cartesian Formulation, Lagrangian Brackets, and Specific Formulae

Let us begin by writing equations (9.1.13) in component form. Assume that the Cartesian components of $\vec{r}$ are x_i. Then equations (9.1.13) become

$$\left.\begin{aligned} \sum_{j=1}^{6} \frac{\partial \dot{x}_i}{\partial \xi_j}\frac{d\xi_j}{dt} &= \frac{\partial \Psi}{\partial x_i} \qquad i = 1,2,3 \\ \sum_{j=1}^{6} \frac{\partial x_i}{\partial \xi_j}\frac{d\xi_j}{dt} &= 0 \qquad\quad i = 1,2,3 \end{aligned}\right\} \quad . \tag{9.2.1}$$

However, the dependence of ξ_j on time is buried in these equations and it would be useful to be able to write them so that $d\vec{\xi}/dt$ is explicitly displayed. To accomplish this multiply each of the first set of equations by $\partial x_i/\partial \xi_k$ and add the three component equations together. This yields:

$$\sum_{i=1}^{3} \frac{\partial x_i}{\partial \xi_k} \sum_{j=1}^{6} \frac{\partial \dot{x}_i}{\partial \xi_j}\frac{d\xi_j}{dt} = \sum_{i=1}^{3} \frac{\partial x_i}{\partial \xi_k}\frac{\partial \Psi}{\partial x_i} = \frac{\partial \Psi}{\partial \xi_k} \qquad k = 1,2,\cdots,6 \quad . \tag{9.2.2}$$

Multiply each of the second set of equations by $(-\partial \dot{x}_i/\partial \xi_k)$ and add them together to get

$$-\sum_{i=1}^{3} \frac{\partial \dot{x}_i}{\partial \xi_k} \sum_{j=1}^{6} \frac{\partial x_i}{\partial \xi_j}\frac{d\xi_j}{dt} = 0 \qquad k = 1,2,\cdots,6 \quad . \tag{9.2.3}$$

Finally add equation (9.2.2) and equation (9.2.3) together, rearrange the order of summation factoring out the desired quantity ($d\xi_j/dt$) to get

$$\sum_{j=1}^{6}\left(\frac{d\xi_j}{dt}\right)\sum_{i=1}^{3}\left[\frac{\partial x_i}{\partial \xi_k}\frac{\partial \dot{x}_i}{\partial \xi_j}-\frac{\partial x_i}{\partial \xi_j}\frac{\partial \dot{x}_i}{\partial \xi_k}\right]=\frac{\partial \Psi}{\partial \xi_k} \quad , \qquad k=1,2,\cdots,6 \quad . \qquad (9.2.4)$$

The ugly looking term under the second summation sign is known as the *Lagrangian bracket* of ξ_k and ξ_j and, by convention, is written as

$$\left[\xi_k,\xi_j\right] \equiv \sum_{i=1}^{3}\left[\frac{\partial x_i}{\partial \xi_k}\frac{\partial \dot{x}_i}{\partial \xi_j}-\frac{\partial x_i}{\partial \xi_j}\frac{\partial \dot{x}_i}{\partial \xi_k}\right] \quad . \qquad (9.2.5)$$

The reason for pursuing this apparently complicating procedure is that the Lagrangian brackets have no explicit time dependence so that they represent a set of coefficients that simply multiply the time derivatives of ξ_j. This reduces the equations of motion to six first order linear differential equations which are

$$\sum_{j=1}^{6}\left[\xi_k,\xi_j\right]\frac{d\xi_j}{dt}=\frac{\partial \Psi}{\partial \xi_k} \qquad k=1,2,\cdots,6 \quad . \qquad (9.2.6)$$

All we need to do is determine the Lagrangian brackets for an explicit set of orbital elements and since they are time independent, they may be evaluated at any convenient time such as t_o .

If we require that the scalar (dot) product be taken over coordinate $(\vec{r})$ space rather than orbital element $(\vec{\xi})$ space, we can write the Lagrangian bracket as

$$\left[\xi_k,\xi_j\right] \equiv \left[\vec{\xi},\vec{\xi}\right] = \left[\frac{\partial \vec{r}}{\partial \vec{\xi}}\bullet\frac{\partial \dot{\vec{r}}}{\partial \vec{\xi}}-\frac{\partial \dot{\vec{r}}}{\partial \vec{\xi}}\bullet\frac{\partial \vec{r}}{\partial \vec{\xi}}\right] \quad . \qquad (9.2.7)$$

Since the partial derivatives are tensors, the scalar product in coordinate space does not commute. However, we may show the lack of explicit time dependence of the Lagrange bracket by direct partial differentiation with respect to time so that

$$\frac{\partial}{\partial t}\left[\vec{\xi},\vec{\xi}\right]=\left[\frac{\partial^2\vec{r}}{\partial t\partial\vec{\xi}}\bullet\frac{\partial\dot{\vec{r}}}{\partial\vec{\xi}}+\frac{\partial\vec{r}}{\partial\vec{\xi}}\bullet\frac{\partial^2\dot{\vec{r}}}{\partial t\partial\vec{\xi}}\right]-\left[\frac{\partial^2\dot{\vec{r}}}{\partial t\partial\vec{\xi}}\bullet\frac{\partial\vec{r}}{\partial\vec{\xi}}+\frac{\partial\dot{\vec{r}}}{\partial\vec{\xi}}\bullet\frac{\partial^2\vec{r}}{\partial t\partial\vec{\xi}}\right] \quad , \tag{9.2.8}$$

or re-arranging the order of differentiation we get

$$\frac{\partial}{\partial t}\left[\vec{\xi},\vec{\xi}\right]=\frac{\partial}{\partial\vec{\xi}}\left[\frac{\partial\vec{r}}{\partial t}\bullet\frac{\partial\dot{\vec{r}}}{\partial\vec{\xi}}-\frac{\partial\vec{r}}{\partial\vec{\xi}}\bullet\frac{\partial\dot{\vec{r}}}{\partial t}\right]-\frac{\partial}{\partial\vec{\xi}}\left[\frac{\partial\dot{\vec{r}}}{\partial\vec{\xi}}\bullet\frac{\partial\vec{r}}{\partial t}-\frac{\partial\vec{r}}{\partial\vec{\xi}}\bullet\frac{\partial\dot{\vec{r}}}{\partial t}\right] \quad . \tag{9.2.9}$$

Using equation (9.1.6) and Newton's laws we can write this as

$$\frac{\partial}{\partial t}\left[\vec{\xi},\vec{\xi}\right]=\frac{\partial}{\partial\vec{\xi}}\left[\frac{1}{2}\frac{\partial v^2}{\partial\vec{\xi}}-\frac{\partial\Phi}{\partial\vec{\xi}}\right]-\frac{\partial}{\partial\vec{\xi}}\left[\frac{1}{2}\frac{\partial v^2}{\partial\vec{\xi}}-\frac{\partial\Phi}{\partial\vec{r}}\bullet\frac{\partial\vec{r}}{\partial\vec{\xi}}\right]=0 \quad . \tag{9.2.10}$$

Remember that we wrote $\vec{r}(\vec{\xi},t)$ so that the coordinates x_i and their time derivatives $\dot{x}_i$ depend only on the set of orbital elements ξ_j and time. Thus the Lagrangian brackets depend only on the particular set of orbital elements and may be computed once and for all. There are various procedures for doing this, some of which are tedious and some of which are clever, but all of which are relatively long. For example, one can calculate them for $t = T_o$ so that $M = E = \nu = 0$. In addition, while one can formulate 36 values of $[\xi_k , \xi_j]$ it is clear from equation (9.2.5) that

$$\left.\begin{aligned}&[\xi_k,\xi_j]=-[\xi_j,\xi_k]\\&[\xi_k,\xi_k]=0\end{aligned}\right\} \quad . \tag{9.2.11}$$

This reduces the number of linearly independent values of $[\xi_k , \xi_j]$ to 15. However, of these 15 Lagrange brackets, only 6 are nonzero [see Taff[11] p.306, 307] and are given below.

$$\left.\begin{aligned}&[i,\Omega]=na^2(1-e^2)^{1/2}\sin i\\&[a,\Omega]=-\tfrac{1}{2}na(1-e^2)\cos i\\&[e,\Omega]=na^2e(1-e^2)^{-1/2}\cos i\\&[a,\mathbf{o}]=-\tfrac{1}{2}na(1-e^2)^{1/2}\\&[e,\mathbf{o}]=na^2e(1-e)^{-1/2}\\&[a,T_0]=\tfrac{1}{2}a\end{aligned}\right\} \quad , \tag{9.2.12}$$

where n is just the mean angular motion given in terms of the mean anomaly M by

$$M = n(t - T_0) \quad . \tag{9.2.13}$$

Thus the coefficients of the time derivatives of ξ_j are explicitly determined in terms of the orbital elements of the osculating orbit $\vec{\xi}_0$.

To complete the solution, we must deal with the right hand side of equation (9.2.6). Unfortunately, the partial derivatives of the perturbing potential are likely to involve the orbital elements in a complicated fashion. However, we must say something about the perturbing potential or the problem cannot be solved. Therefore, let us assume that the behavior of the potential is understood in a cylindrical coordinate frame with radial, azimuthal, and vertical coordinates (r, ϑ, and z) respectively. We will then assume that the cylindrical components of the perturbing force are known and given by

$$\left.\begin{aligned} \Re &\equiv \frac{\partial \Psi}{\partial r} \\ \Im &\equiv \frac{1}{r}\frac{\partial \Psi}{\partial \vartheta} \\ \aleph &\equiv \frac{\partial \Psi}{\partial z} \end{aligned}\right\} \quad . \tag{9.2.14}$$

Then from the chain rule

$$\frac{\partial \Psi}{\partial \xi_i} = \frac{\partial \Psi}{\partial r}\frac{\partial r}{\partial \xi_i} + \frac{\partial \Psi}{\partial \vartheta}\frac{\partial \vartheta}{\partial \xi_i} + \frac{\partial \Psi}{\partial z}\frac{\partial z}{\partial \xi_i} \quad . \tag{9.2.15}$$

The partial derivatives of the cylindrical coordinates with respect to the orbital elements may be calculated directly and the equations for the time derivatives of the orbital elements [equations (9.2.6)] solved explicitly. The algebra is long and tedious but relatively straight forward and one gets

$$
\left.
\begin{aligned}
\frac{da}{dt} &= \frac{2}{n}(1-e^2)^{1/2}\left[\Re e \sin\nu + \Im a(1-e^2)/r\right] \\
\frac{de}{dt} &= \frac{1}{na}(1-e^2)^{1/2}\left[\Re \sin\nu + \Im(\cos E + \cos\nu)\right] \\
\frac{d\,i}{dt} &= \left[na^2(1-e^2)^{1/2}\right]^{-1} \aleph r\cos(\nu + \mathbf{o}) \\
\frac{d\Omega}{dt} &= \left[na^2(1-e^2)^{1/2}\sin i\right]^{-1} \aleph r\sin(\nu + \mathbf{o}) \\
\frac{d\,\mathbf{o}}{dt} &= \frac{(1-e^2)^{1/2}}{nae}\left[\Im\left(\frac{\sin\nu[r + a(1-e^2)]}{a(1-e^2)}\right) - \Re\cos\nu\right] \\
&\qquad + \frac{d\Omega}{dt}\left[2(1-e^2)^{1/2}\sin^2\left(\tfrac{i}{2}\right) - 1\right] \\
\frac{dT_0}{dt} &= \frac{1}{n}\left\{
\begin{aligned}
&\frac{d\Omega}{dt}\left[1 + 2(1-e^2)^{1/2}\sin^2\left(\tfrac{i}{2}\right)\right] - \left[\frac{d\Omega}{dt} - \frac{d\,\mathbf{o}}{dt}\right]e^2\left[1 + (1-e^2)^{1/2}\right]^{-1} \\
&\qquad - \left[\frac{2r\Re}{na^2}\right]
\end{aligned}
\right\}
\end{aligned}
\right\} . \quad (9.2.16)
$$

An alternative set of perturbation equations attributed to Gauss and given by Taff[11] (p.314) is

$$
\left.
\begin{aligned}
\frac{da}{dt} &= \frac{2e\sin\nu}{n(1-e^2)^{1/2}}\Re + \frac{2a(1-e^2)^{1/2}}{nr}\Im \\
\frac{de}{dt} &= \frac{(1-e^2)^{1/2}\sin\nu}{na}\Re + \frac{(1-e^2)^{1/2}}{na^2e}\left[\frac{a^2(1-e^2)-r^2}{r^2}\right]\Im \\
\frac{d\boldsymbol{\omega}}{dt} &= -\frac{(1-e^2)^{1/2}\cos\nu}{nae}\Re + \frac{(1-e^2)^{1/2}\sin\nu}{nae}\left[\frac{a(1-e^2)+r}{a(1-e^2)}\right]\Im \\
&\qquad\qquad - \frac{r\sin(\Omega+\boldsymbol{\omega})\cot i}{L}\aleph \\
\frac{di}{dt} &= \frac{r\cos(\Omega+\boldsymbol{\omega})}{L}\aleph \\
\frac{d\Omega}{dt} &= \frac{r\csc i\,\sin(\Omega+\boldsymbol{\omega})}{L}\aleph \\
\frac{dT_0}{dt} &= +\frac{1}{n^2a}\left[\frac{2r}{a}-\frac{(1-e^2)\cos\nu}{e}\right]\Re + \frac{(1-e^2)\sin\nu}{na^2e}\left[\frac{a(1-e^2)+r}{a(1-e^2)}\right]\Im
\end{aligned}
\right\} . \quad (9.2.17)
$$

where

$$
L = \frac{2\pi a^2}{P} \quad . \qquad (9.2.18)
$$

These relatively complicated forms for the solution show the degree of complexity introduced by the nonlinearity of the equations of motion. However, they are sufficient to demonstrate that the problem does indeed have a solution. Given the perturbing potential and an approximate two body solution at some epoch t_0, one can use all of the two body mechanics developed in previous chapters to calculate the quantities on the right hand side of equations (9.2.16). This allows for a new set of orbital elements to be calculated and the motion of the objects followed in time. The process may be repeated allowing for the cumulative effects of the perturbation to be included.

However, one usually relies on the original assumption that the perturbing forces are small compared to those that produce the two body motion [equation (9.1.3)] .Then all the terms on the left hand side of equation (9.2.16) will be small and the motion can be followed for many orbits before it is necessary to change the orbital elements. That is the major thrust of perturbation theory. It tells you how things ought to change in response to known forces. Thus, if the source of the perturbation lies in the plane defining the cylindrical coordinate system (and the plane defining the orbital elements) $\aleph$ will be zero and the orbital inclination (i) and the longitude of the ascending node will not change in time. Similarly if the source lies along the z-axis of the system, the semi-major axis (a) and eccentricity (e) will be time independent. If the changes in the orbital elements are sufficiently small so that one may average over an orbit without any significant change, then many of the perturbations vanish. In any event, such an averaging procedure may be used to determine equations for the slow change of the orbital elements.

Utility of the development of these perturbation equations relies on the approximation made in equation (9.1.3). That is, the equations are essentially first order in the perturbing potential. Attempts to include higher order terms have generally led to disaster. The problem is basically that the equations of classical mechanics are nonlinear and that the object of interest is $\vec{r}(t)$. Many small errors can propagate through the procedures for finding the orbital elements and then to the position vector itself. Since the equations are nonlinear, the propagation is nonlinear. In general, perturbation theory has not been terribly successful in solving problems of celestial mechanics. So the current approach is generally to solve the Newtonian equations of motion directly using numerical techniques. Awkward as this approach is, it has had great success in solving specific problems as is evidenced by the space program. The ability to send a rocket on a complicated trajectory through the satellite system of Jupiter is ample proof of that. However, one gains little general insight into the effects of perturbing potentials from single numerical solutions.

Problems such as the Kirkwood Gaps and the structure of the Saturnian ring system offer ample evidence of problems that remain unsolved by classical celestial mechanics. However, in the case of the former, much light has been shed through the dynamics of Chaos (see Wisdom[10]). There remains much to be solved in celestial mechanics and the basic nonlinearity of the equations of motion will guarantee that the solutions will not come easily.

Formal perturbation theory provides a nice adjunct to the formal theory of celestial mechanics as it shows the potential power of various techniques of classical mechanics in dealing with problems of orbital motion. Because of the nonlinearity of the Newtonian equations of motion, the solution to even the simplest problem can become very involved. Nevertheless, the majority of dynamics problems involving a few objects can be solved one way or another. Perhaps it is because of this non-linearity that so many different areas of mathematics and physics must be brought together in order to solve these problems. At any rate celestial mechanics provides a challenging training field for students of mathematical physics to apply what they know.

Chapter 9: Exercises

1. If the semi-major axis of a planets orbit is changed by Δa, how does the period change? How does a change in the orbital eccentricity Δe affect the period?

2. If v_1 and v_2 are the velocities of a planet at perihelion and aphelion respectively, show that

$$(1-e)v_1 = (1+e)v_2 \quad .$$

3. Find the Lagrangian bracket for $[e, \Omega]$.

4. Using the Lagrangian and Gaussian perturbation equations, find the behavior of the orbital elements for a perturbative potential that has a pure r-dependence and is located at the origin of the coordinate system.