11

Environment of the Radiation Field

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Thus far, we have said little or nothing about the gas through which the radiation is flowing. This constitutes the second half of the problem of constructing a model for the atmosphere of a star. To get the solutions for the radiation field described in Chapter 10, we must know the opacity, – and not just the mean opacity that was required for stellar interiors, but the frequency-dependent opacity that determines which photons will escape from the star and from what location. We must understand enough of the physics of the gas for that opacity to be determined. Given that, we can calculate the emergent stellar spectrum by solving the equation of radiative transfer. This seems like a small requirement, but as we look more closely at the details of the opacity, the more specific knowledge of the state of the gas is required. For the construction of a model of the atmospheric structure, a little more is required. We must know how the particles that make up the gas interact with each other as well as with photons. Fortunately, with the aid of the assumption of local thermodynamic equilibrium (LTE), much of our task has already been accomplished.

11.1 Statistics of the Gas and the Equation of State

For normal stellar atmospheres, the effective temperatures range from a few thousand degrees to perhaps 50,000 K. The pressures are such as to permit the existence of spectral lines and so the densities cannot be great enough to cause departures from the ideal-gas law. The assumption of LTE implies that the material components of the gas making up the stellar atmosphere behave as if they were in thermodynamic equilibrium characterized by the local value of the kinetic temperature. In Chapter 1 [equations (1.1.16) and (1.1.17)] we found that as long as the density of available cells in phase space is much greater than the particle phase density, such a gas should obey Maxwell-Boltzmann statistics. If this is true, the fraction of particles that have a certain kinetic energy w_i is

$$\frac{N_i}{N} = \frac{g_i e^{-w_i/(kT)}}{U(T)} \tag{11.1.1}$$

Since the pressure is the second velocity moment of the density, the Maxwell-Boltzmann distribution formula [equation (11.1.1)] leads to the ideal-gas law [equation (1.3.4)], namely,

$$P_g = nkT \tag{11.1.2}$$

Later [equation (4.1.6)] we found it convenient to represent the total number of particles in terms of the density and the corresponding mass of hydrogen atoms μm_h that would yield the required number of particles, so that

$$P_g = \frac{\rho kT}{\mu m_h} \tag{11.1.3}$$

The parameter μ is called the *mean molecular weight.*

a Boltzmann Excitation Formula

Under the assumption of LTE, the energy distribution of all particles represents the most probable macrostate for the system. This state is arrived at through random collisions between the gas particles themselves. Such a gas is said to be *collisionally relaxed* and is in stationary equilibrium. This means that all aspects of the gas will exhibit the same energy distribution, including those energy aspects of the gas which do not allow for a continuum distribution of energy states – specifically those states described by the orbital electrons. Thus, an ensemble of atoms will exhibit a distribution of states of electronic excitation that follows the Maxwell-Boltzmann distribution law. Were the energy not shared between the excitation energy and the kinetic energy of the particles, collisions would ensure that energy differences were made up in the deficit population at the expense of the population that had the relative surplus.

Such a situation would not represent a time-independent distribution until equilibrium between the two populations was established and therefore would not be the most probable macrostate. Since the various states of atomic excitation will be distributed according to the Maxwell-Boltzmann distribution law, the number of particles in any particular state of excitation will be

$$\frac{N_j}{N} = \frac{g_j e^{-\epsilon_j/(kT)}}{U(T)} \tag{11.1.4}$$

Here, g_j is the statistical weight, and it has the same meaning as it did in Chapter 1 [equation (1.1.17)]. The parameter ε_j is the excitation energy above the ground state, and U(T) is the partition function. U(T) is nothing more than a normalization parameter that reflects the total number of particles available for distribution among the various energy states. It also has the same meaning as it did for the continuum distribution of energies discussed in Chapter 1, but now will be determined from the sum over the discrete states of excitation. Its role as a normalization parameter of the distribution is most clearly demonstrated by summing equation (11.1.4) over all particles and their energy states so that the left-hand side is unity. This then confirms the form of the partition function as

$$U(T) = \sum_j g_j e^{-\epsilon_j/(kT)} \tag{11.1.5}$$

b Saha Ionization Equilibrium Equation

A completely rigorous derivation of the Saha equation from first principles is long and not particularly illuminating. So instead of performing such a derivation, we appeal to arguments similar to those for the Boltzmann excitation formula. The object now is to find the equilibrium distribution formula for the distribution of the various states of ionization for a collection of atoms. Again, we assume that a time-independent equilibrium exists between the electrons and the ions. However, now the electron population will depend on the equilibrium established for all elements, which appears to make this case quite different from the Boltzmann excitation formula. However, we proceed in a manner similar to that for the Boltzmann excitation formula. Let us try to find the probability of excitation, not for a bound state, but of a state in the continuum where the electron can be regarded as a free particle.

Consider an atom in a particular state of ionization, and denote the number of such atoms in a particular state of excitation, say the ground state, by the quantity n_{00}. The number of these atoms excited to the ionized state where the electron can be regarded as a free particle is then given by the Boltzmann excitation formula as

$$\frac{n_{01}}{n_{00}} = \frac{g_f}{g_{00}} e^{-(\chi_0 + \frac{1}{2}mv^2)/(kT)} \tag{11.1.6}$$

Here g_f is the statistical weight of the final state of the ionized atom, and χ_0 is the ionization potential of the atom in question. The second term in the exponential is simply the energy of the electron that has been elevated to the continuum. Now we can relate the total number of ionized atoms in the ground state to the total number of ionized atoms through the repeated use of the Boltzmann excitation formula and the partition function as

$$n_{0i} = \frac{N_i g_{0i} e^{-\epsilon_0/(kT)}}{U(T)} \tag{11.1.7}$$

where N_i is the total number of atoms in the ith state of ionization. However, since the resultant ionized atoms we are considering are in their ground state, ε_0 is zero by definition. In addition, the statistical weight of the final state g_f, can be written as the product of the statistical weight of the ground state of the ionized atom and that of a free electron. We may now use this result and equation (11.1.6) to write

$$\frac{n_{01}}{n_{00}} = \frac{N_1 g_{01}/U_1(T)}{N_0 g_{00}/U_0(T)} = \frac{g_{01} g_e}{g_{00}} e^{-(\chi_0 + \frac{1}{2}mv^2)/(kT)} \tag{11.1.8}$$

which simplifies to

$$\frac{N_1}{N_0} = \frac{U_1(T)}{U_0(T)} e^{-\chi_0/(kT)} g_e e^{-\frac{1}{2}mv^2/(kT)} \tag{11.1.9}$$

Although we picked a specific state of excitation – the ground state – to arrive at equation (11.1.9), that choice was in no way required. It only provided us with a way to use the Boltzmann excitation formula for atoms in two differing states of ionization.

The statistical weight of a free electron is really nothing more than the probability of finding a given electron in a specific cell of phase space, so that

$$g_e = \frac{2\, dx\, dy\, dz\, dp_x\, dp_y\, dp_z}{h^3} \tag{11.1.10}$$

Again, for electrons, the familiar factor of 2 arises because the spin of an electron can be either "up" or "down". Assuming that the microscopic velocity field is isotropic, we can replace the "momentum volume" by its spherical counterpart and express it in terms of the velocity:

$$dp_x\, dp_y\, dp_z = 4\pi p^2\, dp = 4\pi m^3 v^2\, dv \tag{11.1.11}$$

The "space volume" of phase space occupied by the electron can be expressed in terms of the inverse of the electron number density, so that the statistical weight of an electron becomes

$$g_e = \frac{8\pi m^3 v^2\, dv}{N_e h^3} \tag{11.1.12}$$

and equation (11.1.9) can be written as

$$\frac{N_e N_1}{N_0} = \frac{U_1(T)}{U_0(T)}\, e^{-\chi_0/(kT)}\, \frac{8\pi m_e^3 v^2}{h^3}\, e^{-\frac{1}{2}m_e v^2/(kT)}\, dv \tag{11.1.13}$$

So far we have assumed that the ionization produced an electron in a specific free state, that is, with a specific velocity in the energy continuum. However, we are interested in only the total number of ionizations, so we must integrate equation (11.1.13) over all allowed velocities for the electrons that result from the ionization process. Thus,

$$\frac{N_e N_1}{N_0} = \frac{U_1(T)}{U_0(T)}\, e^{-\chi_0/(kT)}\, \frac{8\pi m_e^3 v^2}{h^3}\, e^{-\frac{1}{2}m_e v^2/(kT)}\, dv \tag{11.1.14}$$

For convenience, we also assumed that the states of ionization of interest were the neutral and first states of ionization. However, the argument is correct for any two adjacent states of ionization so we can write with some generality

$$\frac{N_{i+1} N_e}{N_i} = \frac{U_{i+1}(T)}{U_i(T)}\, \frac{2(2\pi k T m_e)^{3/2}}{h^3}\, e^{-\chi_i/(kT)} \tag{11.1.15}$$

This expression is often written in terms of the electron pressure as

$$\frac{N_{i+1}}{N_i} P_e = \frac{2(2\pi m_e)^{3/2}(kT)^{5/2}}{h^3}\, \frac{U_{i+1}(T)}{U_i(T)}\, e^{-\chi_i/(kT)} \tag{11.1.16}$$

Both expressions [equations (11.1.15) and (11.1.16)] are known as the *Saha ionization equation*. Its validity rests on the Boltzmann excitation formula and the velocity distribution of the electrons produced by the ionization being described by the Boltzmann distribution formula. Both these conditions are met under the conditions of LTE. Indeed, many authors take the validity of the Saha and Boltzmann formulas as a definition for LTE.

The Boltzmann excitation equation and the Saha ionization equation can be combined to yield the fraction of atoms in a particular state of ionization and excitation. Knowing that fraction, we are in a position to describe the extent to which those atoms will impede the flow of photons through the gas.

11.2 Continuous Opacity

In Section 4.1b we discussed the way in which a gas can absorb photons, and we calculated the continuous opacity due to an atom of hydrogen. The calculation of the opacity of other individual species of atoms follows the same type of argument, and so we do not deal with them in detail here since these details can be found in other elsewhere[1]. However, in the stellar interior, we were able to characterize the opacity of the stellar material by a single parameter known as the *Rosseland mean opacity*. This resulted from the fact that the radiation field was itself in thermodynamic equilibrium and therefore depended on the temperature alone. Therefore, all parameters that arise from the interaction of the radiation field and the gas, which is also in thermodynamic equilibrium, must be described in terms of the state variables alone. Unfortunately, in the stellar atmosphere, although the gas can still be considered to be in thermodynamic equilibrium (LTE), the radiation field is not. Thus, the opacity must be determined for each frequency for which a significant amount of radiant energy is flowing through the gas.

Traditionally, the dominant source of opacity has been considered to be that arising from bound-free atomic transitions which are called "continuous" opacity sources. However, contributions to the total opacity that result from bound-bound atomic transitions have been found to play an important role in forming the structure of the atmosphere in a wide variety of stars. Since we will deal with the formation of spectral lines arising from bound-bound transitions in considerable detail later, we defer the discussion of bound-bound opacity until then. It is sufficient to know that the total opacity can be calculated for each frequency of importance and to explicitly consider some of the important sources.

a Hydrogenlike Opacity

Any atom which has a single electron in its outer shell will absorb photons in a manner similar to that for hydrogen, so we should expect the opacity to have a form similar to that of equation (4.1.17). Indeed the expression differs from that of hydrogen by only a factor involving the atomic weight and the atomic constants appropriate for the particular atom. Thus, the opacity per gram of the ionized species is

$$\kappa_\nu(\text{H-like}) = \frac{32\pi e^6 \mathrm{R} Z^4 e^{-\chi_{\mathrm{H1}}/(kT)}}{3\sqrt{3}h^3 c m_h \nu^3}\left(\sum_{n,\nu_n<\nu}^{\infty} \frac{g_n}{n^3} e^{-\epsilon_n/(kT)} + \frac{g_{\mathrm{ff}} kT}{2\chi_{\mathrm{H1}}}\right) , \tag{11.2.1}$$

where R is the Rydberg constant. For ionized helium, Z = 2. Since hydrogen and helium are by far the most abundant elements in stars, neutral hydrogen and ionized helium must be considered major sources of opacity.

b Neutral Helium

Again, because of the large abundance of neutral helium it long been regarded as an important source of stellar opacity. However, since neutral helium has two electrons in the outer shell, the atomic absorption coefficient is much more difficult to calculate. Approximate values for the opacity of neutral helium were first given by Ueno et al.[2] in 1954. Later Stewart and Webb[3] calculated the opacity arising from the ground state, and in general the contributions from the first two excited states must be treated separately. For states of excitation greater than 2, the approximate solutions will generally suffice.

c Quasi-atomic and Molecular States

Although molecular opacity plays an important role in the later-type stars (and will be dealt with later), one might think that molecular opacity is unimportant at temperatures greater than those corresponding to the disassociation energies of the molecules. However, some molecules and atomic states may form for a short time and absorb photons before they disassociate. If the abundance of the atomic species that give rise to these quasi-states is great, they may provide a significant source of opacity. The prototypical example of a short lived or quasi-state is the H-minus ion.

Classically, the existence of the H-minus ion can be inferred from the incomplete screening of the proton by the orbiting electron of Hydrogen (see figure 11.1). Although the orbiting electron is "on one side" of the proton, it is possible to bind an additional electron to the atom for a short time. During this time, the additional electron can undergo bound-free transitions. Quantum mechanically, there exists a single weakly bound state for an electron near a neutral hydrogen atom. This state is weakly bound since the dipole moment of the hydrogen atom is small. The negatively charged configuration is called the H-minus ion and is important in stars only because of the great abundance of hydrogen and electrons at certain temperatures and densities. The Saha equation for the abundance of such ions will be somewhat different from that for normal atoms as the existence of the ion will depend on the availability of electrons as well as hydrogen atoms. Thus the Saha equation for the H-minus ion would have the form

$$\frac{N(\mathrm{H}^-)}{N(\mathrm{H})} = \Phi(T)P_e \tag{11.2.2}$$

The parameter **Φ**(*T*) involves partition functions and the like, but depends on only the temperature. Since the Saha equation for neutral hydrogen implies that the ratio of ionized to neutral hydrogen will also be proportional to the electron pressure, the abundance of the H-minus ion will depend quadratically on the electron pressure. Thus, the relative importance of H-minus to hydrogen opacity will decrease with decreasing pressure. So H-minus is less important than hydrogen as an opacity source for giants than for main sequence dwarfs of the same spectral type.

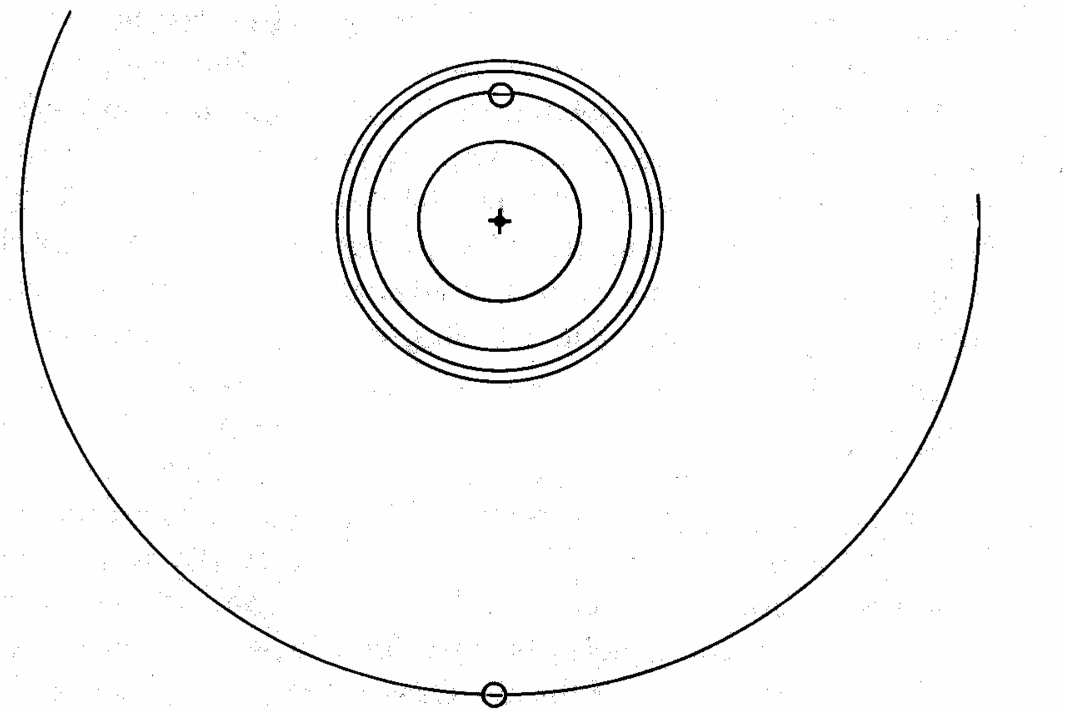

Figure 11.1 shows a classical representation of a Bohr Hydrogen atom with an additional electron bound temporarily so as to form an H-minus ion.

The opacity of a single H-minus ion has been calculated by a number of people over the past 40 years and is not a simple quantum mechanical calculation. The weak binding of the additional electron allows for the existence of a single bound state from which bound-free transitions make contributions to the continuous opacity. The contribution from the bound-free transitions has a peak at about 0.8 micrometers (μm) and is roughly bell-shaped in frequency with a half width of about 1 μm. The free-free contribution rises steadily into the infrared, becoming equal to the bound-free absorption at about 1.45 μm.

d Important Sources of Continuous Opacity for Main Sequence Stars

OB Stars For the hottest stars on the main sequence (the O stars), most of the hydrogen is ionized along with a substantial amount of the helium. The largest source of continuous opacity is therefore due to electron scattering. From spectral type B2 to O, neutral helium joins electron scattering as an important opacity source. In the early-type stars later than B2, hydrogen becomes a significant opacity source which increases in importance as one considers later B-type stars. In these stars, the opacity due to bound-bound (see Section 15.4) and bound-free transitions in the metals becomes increasingly important for the ultraviolet part of the spectrum.

Stars of Spectral Type A0 to F5 At spectral type A0, neutral hydrogen is the dominant source of continuous opacity. As one moves to later spectral types, H-minus opacity increases in importance and dominates the opacity in the late A type and F stars. For stars of spectral type F, another molecule emerges as an important opacity source. Although the H_2 molecule cannot exist for long at these temperatures, enough does exist at any instant so that the once ionized form, H_2^+, provides up to 10 percent of the continuous opacity. The H-minus opacity continues to grow in importance throughout this range of spectral type.

Stars of Spectral Type F5 to G In this range H-minus continues to provide more than 60 percent of the continuous opacity. However, other molecules are not of particular importance until one gets to spectral types later than G. The continuous opacity of metals is particularly important in the ultraviolet range below 3000Å. The opacity due to atomic hydrogen diminishes steadily into the G stars, but is still of major importance.

Late Spectral-Type K to M Stars Very little of the spectral energy distribution of these stars can be considered to result from continuum processes - particularly in the later spectral types. The absorption from the myriads of discrete transitions of molecules so dominates the spectrum of the late-type stars that little is apparent except the large molecular absorption bands. However, insofar as continuum processes still take place, Rayleigh scattering from H_2 molecules is the most important source. Absorption arising from the disassociation of molecules also provides an important source of continuous opacity, particularly in the infrared region of the spectrum.

11.3 Einstein Coefficients and Stimulated Emission

An extremely useful way to view the processes of absorption is to consider the specific types of interactions between a photon and an atom. This view will become the preferred one when we consider problems where we cannot assume LTE. In quantum mechanics, it is customary to think of the probability of the occurrence of a specific event rather than the classical picture of the frequency with which the event occurs. These are really opposite sides of the same coin, but they yield somewhat philosophically different pictures of the processes in question. In considering the transition of an atom from one excited state to another, Einstein defined a set of coefficients that denote the probability of specific transitions taking place. These are known as the *Einstein coefficients*. All radiative processes and the equation of radiative transfer itself may be formulated in terms of these coefficients. The coefficients are determined by the wave functions of the atom alone and thus do not depend in any way on the environment of the atom. (As with most rules, there is an exception. If the density is sufficiently high that the presence of other atoms distorts the wave functions of the atom of interest, then the Einstein coefficients of that atom can be modified.)

Consider an atomic transition which takes an atom from a lower-energy state of excitation n' to a higher-energy state n (or vice versa). The upper state n may be a bound or continuum state, and the transition will involve the absorption or emission of a photon. The number of transitions that will occur in a given time interval for an ensemble of atoms will then depend on the probability that one transition will occur times the number of atoms available to make the transition. There types of transitions can occur:

1. *Spontaneous emission*, where the electron spontaneously makes a downward transition with the accompanying emission of a photon.

2. *Stimulated absorption*, where a passing photon is absorbed, producing the resulting transition.

3. *Stimulated emission*, where the electron makes a downward transition with the accompanying emission of a photon. If this occurs in the presence of a photon of the same type as that emitted by the transition, the probability of the event is greatly enhanced.

The symmetric process of spontaneous absorption simply cannot occur because one cannot absorb what is not there.

We can describe the number of atoms undergoing these processes in terms of the probabilities of the occurrence of a single event:

$$N_{n\to n'} = N_n A_{nn'}\,dt$$
$$N_{n'\to n} = N_{n'} B_{n'n} I(\nu_{nn'})\,dt$$
$$N_{n\to n'} = N_n B_{nn'} I(\nu_{n'n})\,dt \tag{11.3.1}$$

The dependence of the stimulated processes on the presence of photons is made clear by the inclusion of the specific intensity, corresponding to the energy of the transition, in the last two equations. The coefficients that appear in these equations are known as the Einstein coefficient of spontaneous emission $A_{nn'}$, the Einstein coefficient of absorption (stimulated) $B_{n'n}$, and the Einstein coefficient of stimulated emission $B_{nn'}$. Since there is only one kind of absorption processes, the adjective stimulated is usually dropped from the coefficient $B_{n'n}$.

a Relations among Einstein Coefficients

The three Einstein coefficients are not linearly independent. Indeed, the specification of any one of them allows the determination of the other two. To show this, we construct an environment where we know something about the rates at which processes should take place. Since the Einstein coefficients are atomic constants, they are independent of the environment, and thus we are free to create any environment that we choose as long as it is physically self-consistent. With this in mind, consider a gas that is in STE. Under these conditions, the atomic transition rates in and out of each level are equal (detailed balancing). If this were not the case, cyclical processes could exist that would provide for a flow of energy from one frequency to another. But the assumption of STE requires that the photon energy distribution be given by the Planck function and that situation would not be preserved by an energy flow in frequency space. Therefore, it can not happen in STE. In addition, in STE the Boltzmann excitation formula holds for the distribution of atoms among the various states of excitation. Thus, the second of equations (11.3.1) must be equal to the sum of the other two.

$$N_{n'} B_{n'n} B_\nu(T) = N_n[A_{nn'} + B_{nn'} B_\nu(T)] \tag{11.3.2}$$

But since the Boltzmann formula must hold,

$$\frac{N_n}{N_{n'}} = \frac{g_n}{g_{n'}} e^{-h\nu_{nn'}/(kT)} \tag{11.3.3}$$

Substituting in the correct form for the Planck function [see equation (1.1.24)] and noting that the frequency ν is the same as the frequency that appears in the excitation energy of the Boltzmann formula $\nu_{nn'}$ we get

$$A_{nn'}\frac{g_n}{g_{n'}} = \frac{2h\nu^3}{c^2} B_{n'n} \frac{e^{h\nu/(kT)} - B_{nn'}g_n/(B_{n'n}g_{n'})}{e^{h\nu/(kT)} - 1}$$

(11.3.4)

Now the Einstein coefficients are atomic constants and therefore must be independent of the temperature. This can happen only if the numerator of the rightmost fraction in equation 11.3.4 is identically 1. Thus,

$$\frac{B_{nn'}g_n}{B_{n'n}g_{n'}} = 1 \qquad A_{nn'} = \left(\frac{2h\nu^3}{c^2}\frac{g_{n'}}{g_n}\right)B_{n'n}$$

(11.3.5)

These relationships must be completely general since the Einstein coefficients must be independent of the environment.

b Correction of the Mass Absorption Coefficient for Stimulated Emission

In deriving the equation of radiative transfer, we took no notice of the concept of stimulated emission. The mass emission coefficient j_ν implicitly contains the notion since it represents the total energy emitted per gram of stellar material. However, the mass absorption coefficient κ_ν was calculated as the effective cross section per gram of stellar material and thus counts only those photons absorbed. Should a passing photon stimulate the production of an additional photon, that processes should be counted as a "negative" absorption. Indeed, some authors call the coefficient of stimulated emission the *coefficient of negative absorption*. Now it is a property of the stimulated emission process that the photon produced by the passage of another photon has exactly the same direction, energy, and phase. Indeed, this is the mechanism by which lasers work and which we discuss in greater depth in the chapters dealing with line formation. To correct the absorption coefficient for the phenomenon of stimulated emission, we need only conserve energy.

Consider the total energy produced within a cubic centimeter of a star and flowing into a solid angle d**Ω**.

$$j_\nu\rho\, d\nu\, d\Omega = h\nu N_n(A_{nn'} + B_{nn'}I_\nu) = N_n A_{nn'} h\nu\left(1 + \frac{c^2 I_\nu}{2h\nu^3}\right)$$

(11.3.6)

This must be equal to the total energy absorbed in that same cubic centimeter from the same solid angle.

$$I_\nu\kappa_\nu\rho\, d\nu\, d\Omega = N_{n'}B_{n'n}I_\nu h\nu$$

(11.3.7)

Now consider an environment that is in LTE and in which scattering is unimportant. The source function for such an atmosphere is then

$$S_\nu(\tau_\nu) = \frac{j_\nu}{\kappa_\nu} = \frac{N_n}{N_{n'}} \frac{2h\nu^3}{c^2} \frac{g_{n'}}{g_n} \left(\frac{1 + c^2 I_\nu}{2h\nu^3} \right)$$

$$S_\nu = B_\nu(1 - e^{-h\nu/(kT)}) + I_\nu e^{-h\nu/(kT)} \tag{11.3.8}$$

Now, if we insert this form for the source function into the plane-parallel equation of radiative transport, we get

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu = (I_\nu - B_\nu)(1 - e^{-h\nu/(kT)}) \tag{11.3.9}$$

However, our original equation for this problem had the form

$$\mu \frac{dI_\nu}{d\tau'_\nu} = I_\nu - B_\nu \tag{11.3.10}$$

We can bring equation (11.3.9) into the same form as equation (11.3.10) by simply redefining the mass absorption coefficient κ_ν as

$$\kappa'_\nu = \kappa_\nu(1 - e^{-h\nu/(kT)}) \qquad d\tau'_\nu = -\kappa'_\nu \rho \, dx \tag{11.3.11}$$

Modifying the mass absorption coefficient by the factor $1\text{-}e^{h\nu/kT}$ simply corrects it for the effects of stimulated emission. Once again we are correcting atomic parameters that are independent of their environment and so the result is independent of the details of the derivation. That is, the correction term is a general one and applies to all problems of radiative transfer. Thus, one should always be sure that the absorption coefficients used are corrected for stimulated emission. This is particularly true when one is using tabular opacities that are generated by someone else.

11.4 Definitions and Origins of Mean Opacities

In Chapter 4, we averaged the frequency-dependent opacity over wavelength in order to obtain the Rosseland mean opacity. We claimed that this was indeed the correct opacity to use in the case of STE. That such a mean should exist in the case of STE seemed reasonable since the fundamental parameters governing the structure of the gas should depend on only the temperature. That the appropriate average should be the Rosseland mean is less obvious. In the early days of the study of stellar atmospheres a considerable effort was devoted to reducing the nongray problem of radiative transfer to the gray problem, because the gray atmosphere had been well studied and there were numerous methods for its description. The general idea was that there should exist some "mean" opacity that would reduce the problem to the gray problem or perhaps one that was nearly gray. We know now that such a mean does not exist, but the arguments used in the search are useful to review if for no other reason than many of the mean opacities that were proposed can still be found in the literature and often have some utility in describing the properties of an atmosphere. They are often used for calculating optical depths which label various depth points in the atmosphere. For a particularly good discussion of mean opacities, see Mihalas[4].

Consider the equation of radiative transfer for a plane-parallel atmosphere and its first two spatial moments. Table 11.1 contains these expressions for both the gray and nongray case. Ideally, we would like to define a mean opacity so that all the moment equations for the nongray case look mathematically like those for the gray case.

a Flux-Weighted (Chandrasekhar) Mean Opacity

Suppose we choose a mean so that the last of the moment equations takes on the gray form. We can obtain such a mean by

$$-\frac{1}{\rho}\frac{d}{dx}\int_0^\infty K_\nu\,d\nu = -\frac{1}{\rho}\frac{dK}{dx} = \frac{1}{4}\int_0^\infty \kappa_\nu F_\nu\,d\nu \equiv \frac{1}{4}\langle\kappa_\nu\rangle_F F \tag{11.4.1}$$

so that the mean is defined by

$$\langle\kappa_\nu\rangle_F = \int_0^\infty \frac{\kappa_\nu F_\nu\,d\nu}{F} \tag{11.4.2}$$

The use of such a mean will indeed reduce the nongray equation for the radiation pressure gradient (3) to the gray form. Such a mean opacity is often referred to as a *flux-weighted mean*, or the *Chandrasekhar mean opacity*. Unfortunately, such a mean will not reduce either of the other two moment equations from the nongray to the gray case. However, it does yield a simple expression for the radiation pressure gradient which is useful for high-temperature atmospheres where the radiation pressure contributes significantly to the support of the atmosphere, namely,

$$\frac{dP_r}{d\tau_F} = \frac{\sigma T_{\mathrm{eff}}^4}{c} \qquad d\tau_F = -\langle\kappa_\nu\rangle_F\rho\,dx \tag{11.4.3}$$

b Rosseland Mean Opacity

Suppose we require of the third equation in Table 11.1 that

$$\int_0^\infty F_\nu\,d\nu = F \tag{11.4.4}$$

This quite reasonable requirement means that the nongray moment equation will become

$$\int_0^\infty \frac{1}{\rho\kappa_\nu}\frac{dK_\nu}{dx}\,d\nu = -\frac{1}{4}\int_0^\infty F_\nu\,d\nu = -\frac{F}{4} \equiv \frac{1}{\rho\langle\kappa_\nu\rangle_R}\int_0^\infty \frac{dK_\nu}{dx}\,d\nu \tag{11.4.5}$$

implies that the mean opacity has the following form:

$$\frac{1}{\langle\kappa_\nu\rangle} = \frac{\int_0^\infty (1/\kappa_\nu)(dK_\nu/dx)\,d\nu}{\int_0^\infty (dK_\nu/dx)\,d\nu} \tag{11.4.6}$$

Table 11.1 Equations of Radiative Transfer for a Plane-Parallel Atmosphere

	Gray Atmosphere	Non-gray Atmosphere
(1)	$\mu\frac{dI}{d\tau} = I - J$	$\mu\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$
(2)	$\frac{dF}{d\tau} = 0$	$\frac{dF_\nu}{d\tau_\nu} = 4(J_\nu - S_\nu)$
(3)	$\frac{dK}{d\tau} = \frac{F}{4}$	$\frac{dK_\nu}{d\tau_\nu} = \frac{F_\nu}{4}$

As one moves more deeply into the star, the near isotropy of the radiation field and approach to STE will require that

$$K_\nu \to \frac{J_\nu}{3} \to \frac{B_\nu(t)}{3} \tag{11.4.7}$$

Since the Planck function depends on the temperature alone,

$$\frac{dB_\nu(T)}{dx} = \frac{\partial B_\nu}{\partial T}\frac{dT}{dx} \tag{11.4.8}$$

Substitution of equations (11.4.7) and (11.4.8) into equation (11.4.6) yields

$$\frac{1}{\langle\kappa_\nu\rangle_R} \equiv \frac{\int_0^\infty (1/\kappa_\nu)\partial B_\nu(T)/\partial T\,d\nu}{\int_0^\infty [\partial B_\nu(T)/\partial T]\,d\nu} \tag{11.4.9}$$

which is identical to equation (4.1.18) for the appropriate mean for stellar interiors. Thus, under the conditions of STE, we find that the Rosseland mean opacity does indeed remove the frequency dependence from the radiative transfer problem. However, in the stellar atmosphere the conditions required by equation (11.4.7) do not apply, so that the Rosseland mean will not fulfill the same function for the theory of stellar atmospheres as it does in the theory of stellar interiors.

c Planck Mean Opacity

Finally, let us consider a mean opacity that will yield a correct value for the thermal emission. Thus,

$$\langle \kappa_\nu \rangle_P \equiv \frac{\int_0^\infty \kappa_\nu B_\nu(T)\, d\nu}{\int_0^\infty B_\nu(T)\, d\nu} \tag{11.4.10}$$

To appreciate the utility of this mean, we develop the condition of radiative equilibrium for the nongray case and see how that approaches the gray result. Let us see what condition would be placed on a mean opacity in order to bring radiative equilibrium in line with the gray case. Consider a mean opacity such that

$$\int_0^\infty (\kappa_\nu - \bar{\kappa}_\nu)[J_\nu(\tau_\nu) - B_\nu(\tau_\nu)]\, d\nu = 0 \tag{11.4.11}$$

This is just a condition that expresses the difference between the nongray condition for radiative equilibrium and the gray condition, both of which should be zero. Now from equation (10.1.13), we can write J_ν as

$$J_\nu(\tau_\nu) = \tfrac{1}{2} \int_0^\infty S_\nu(t) E_1 |t - \tau_\nu|\, dt \tag{11.4.12}$$

Under the condition that $S_\nu(t)=B_\nu[T(t)]$, we can expand the source function in equation (11.4.12) in a Taylor series about τ_ν and integrate term by term, to obtain

$$J_\nu(\tau_\nu) \approx \tfrac{1}{2} B_\nu(\tau_\nu)[2 - E_2(\tau_\nu)] + \cdots + \tag{11.4.13}$$

Thus, near the surface of the atmosphere

$$J_\nu(\tau_\nu) - B_\nu(\tau_\nu) \approx -\tfrac{1}{2} B_\nu(\tau_\nu) \qquad \tau_\nu \ll 1 \tag{11.4.14}$$

Substitution of equation (11.4.14) into equation (11.4.11) yields

$$\bar{\kappa}_\nu = \frac{\int_0^\infty \kappa_\nu B_\nu(T)\, d\nu}{\int_0^\infty B_\nu(T)\, d\nu} \equiv \langle \kappa_\nu \rangle_P \tag{11.4.15}$$

Thus, the Planck mean opacity is the most physically relevant mean for satisfying radiative equilibrium near the surface of the atmosphere.

These various means all have their regions of validity and utility, but none can fulfill the promise of reducing the nongray atmosphere problem to that of a gray atmosphere. Indeed, clearly such a mean does not exist. Three separate moment equations are listed in table 11.1 and a mean opacity represents only one parameter available to bring them all into conformity with their gray analogs. For an arbitrary behavior for κ_ν, this is impossible. Thus, we must be content with solving the nongray radiative transfer problem at all frequencies for which a significant amount of flux flows through the atmosphere.

11.5 Hydrostatic Equilibrium and the Stellar Atmosphere

Any discussion of the environment of the radiation field would be incomplete without some mention of the motions to be expected within the atmosphere. Recently, considerable effort has been expended in describing radiation-driven winds that originate in the outer reaches of the atmospheres of hot stars. This more advanced topic is well beyond the scope of our interest at this point. Instead, we take advantage of the fact that radiation is the primary mode of energy transport through the stellar atmosphere, and we assume that the atmosphere is in hydrostatic equilibrium. While turbulent convection is present in many stars, the motions implied by the observed velocities are likely to cause problems only for the stars with the lowest surface gravities. For stars on the main sequence, hydrostatic equilibrium is an excellent assumption.

The notion of hydrostatic equilibrium has been mentioned repeatedly throughout this book from the general concept obtained from the Boltzmann transport equation in Chapter 1, through the explicit formulation for spherical stars given by equation (2.1.6) to the representation for plane-parallel atmospheres found in equation (9.1.1). However, the introduction of the concept of optical depth allows for a further refinement. Since the logical depth variable for radiative transfer is the optical depth variable τ_ν, it makes some sense to reformulate the other structure equations that explicitly involve the depth coordinate to reflect the use of the optical depth and the independent variable of the structure. Some atmosphere modeling codes use a variant of equation (9.1.1),

$$\frac{1}{\rho}\frac{dP}{dx} = -g \tag{11.5.1}$$

so that the quantity ρdx is the depth variable. A more common formulation takes advantage of the definition of the optical depth

$$d\tau_\nu = -(\kappa_\nu + \sigma_\nu)\rho\, dx \tag{11.5.2}$$

which yields the condition for hydrostatic equilibrium in terms of the optical depth. Thus,

$$\frac{dP}{d\tau_\nu} = \frac{g}{\kappa_\nu + \sigma_\nu} \tag{11.5.3}$$

Having described the environment of the radiation field in terms of the state variables P, T, and ρ, we can now proceed to the actual construction of a model atmosphere.

Problems

1. Find the ratio of ionized to neutral hydrogen at τ_λ (λ = 5000Å) = 1 in (a) the sun, (b) Sirius, and (c) Rigel.

2. Find the ratio of SiI to SiII to SiIII in η Ursa Majoris.

3. Consider an atmosphere made up of pure hydrogen that is almost all in a neutral state. The radiation pressure is negligible, and the opacity is due to the H-minus ion.

   (a) Determine how the pressure at a given optical depth depends on the surface gravity (i.e., how it would change if g were changed).
   (b) Express the pressure as a function of the temperature. Leave the answer in integral form.
   (c) How can one find the pressure as a function of physical depth? Display explicitly all equations.
   (d) Suppose that the effective temperature is T_e and that one wants to find the pressure at the point in the atmosphere corresponding to that temperature. Further, suppose that the surface temperature is in error by 10 percent. Estimate the resultant error in $P(T_e)$.

4 Use a model atmosphere code to construct a model as described in Problem 3 (try T_e = 6000 K and log g = 3 and log g = 4). Compare your results with those of Problem 3.

5. Consider a gas composed of 60 percent hydrogen and 40 percent helium at 10,000 K in a gravity field where log g = 4.5.

   (a) Find the ratio of neutral to ionized hydrogen and the ratio of the three states of ionization for helium.
   (b) Find the ratio of the level populations for the first four levels of excitation in neutral hydrogen.

6. Use a model atmosphere code to find how the state of ionization of hydrogen varies with physical depth in a star with T_e = 10,000 K and Log g = 4.0. Repeat the calculation for a star with T_e= 7000 K and Log g = 1.5. Compare the two cases.

References

1. For additional reading on the subject of stellar atmospheric opacities, consult:

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