

# *Part I*

## *Stellar Interiors*

# 1

## *Introduction and Fundamental Principles*

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The development of a relatively complete picture of the structure and evolution of the stars has been one of the great conceptual accomplishments of the twentieth century. While questions still exist concerning the details of the birth and death of stars, scientists now understand over 90% of a star's life. Furthermore, our understanding of stellar structure has progressed to the point where it can be studied within an axiomatic framework comparable to those of other branches of Physics. It is within this axiomatic framework that we will study stellar structure stellar spectra - the traditional source of virtually all information about stars.

## *1. Introduction and Fundamental Principles*

This book is divided into two parts: stellar interiors and stellar atmospheres. While the division between the two is fairly arbitrary, it is a traditional division separating regimes where different axioms apply. A similar distinction exists between the continuum and lines of a stellar spectrum. These distinctions represent a transition zone where one physical process dominates over another. The transition in nature is never abrupt and represents a difference in degree rather than in kind.

We assume that the readers know what stars are, that is, have a working knowledge of the Hertzsprung-Russell diagram and of how the vast wealth of knowledge contained in it has been acquired. Readers should understand that most stars are basically spherical and should know something about the ranges of masses, radii, and luminosities appropriate for the majority of stars. The relative size and accuracy of the stellar sample upon which this information is based must be understood before a theoretical description of stars can be believed. However, the more we learn about stars, the more the fundamentals of our theoretical descriptions are confirmed. The history of stellar astrophysics in the twentieth century can be likened to that of a photographer steadily sharpening the focus of the camera to capture the basic nature of stars.

In this book, the basic problem of stellar structure under consideration is the determination of the run of physical variables that describe the local properties of stellar material with position in the star. In general, the position in the star is the independent variable(s) in the problem, and other parameters such as the pressure  $P$ , temperature  $T$ , and density  $\rho$  are the dependent variables. Since these parameters describe the state of the material, they are often referred to as state variables. Part I of this book discusses these parameters alone. In Part II, when we arrive near the surface of the star, we shall also be interested in the detailed distribution of the photons, particularly as they leave the star.

Although there are some excursions into the study of nonspherical stars, the main thrust of this book is to provide a basis for understanding the structure of spherical stars. Although the proof is not a simple one, it would be interesting to show that the equilibrium configuration of a gas cloud confined solely by gravity is that of a sphere. However, instead of beginning this book with a lengthy proof, we simply take the result as an axiom that all stars dominated by gravity alone are spherical.

We describe these remarkably stable structures in terms of microphysics, involving particles and photons which are largely in equilibrium. Statistical mechanics is the general area of physics that deals with this subject and contains the axioms that form the basis for stellar astrophysics. Our discussion of stellar structure centers on the interaction of light with matter. We must first describe the properties

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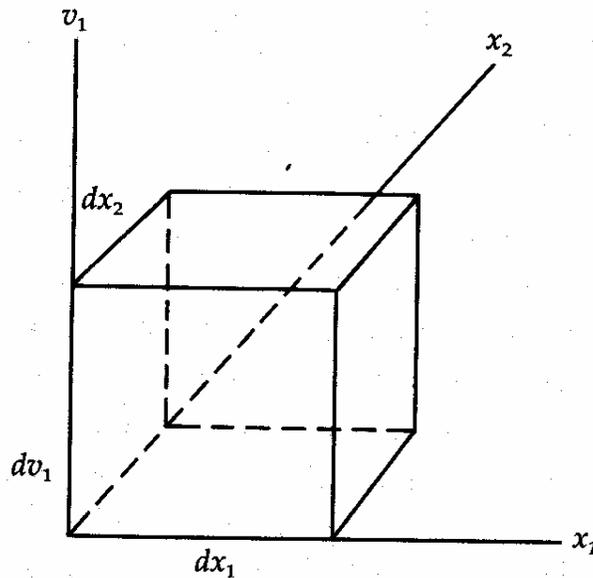
of the space in which the interaction will take place. It is not the normal Euclidean three-dimensional space of intuition, but a higher-dimension space. This higher-dimension space, called phase space, includes the momentum distribution of the particles which make up the star as well as their physical location.

### 1.1 Stationary or "Steady" Properties of Matter

#### a. Phase Space and Phase Density

Consider a volume of physical space that is small compared to the physical system in question, but still large enough to contain a statistically significant number of particles. The range of physical space in which this small volume is embedded may be infinite or finite as long as it is significantly larger than the small volume. First let a set of three Cartesian coordinates  $x_1$ ,  $x_2$ , and  $x_3$  represent the spatial part of the volume. Then allow the additional three Cartesian coordinates  $v_1$ ,  $v_2$ , and  $v_3$  represent the components of the velocity of the particles. Coordinates  $v_1$ ,  $v_2$ , and  $v_3$  are orthogonal to the spatial coordinates. This simply indicates that the velocity and position are assumed to be uncorrelated. It also provides for a six-dimensional space which we call phase space. The volume of the space is

$$dV = dx_1 dx_2 dx_3 dv_1 dv_2 dv_3 \tag{1.1.1}$$



**Figure 1.1** shows part of a small differential volume of phase space. It must be remembered that the position and velocity coordinates are orthogonal to each other.

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If the number of particles in the small volume  $dV$  is  $N$ , then we can define a parameter  $f$ , known as the phase density, by

$$f(x_1, x_2, x_3, v_1, v_2, v_3)dV = N \quad (1.1.2)$$

The manner in which a number of particles can be arranged in an ensemble of phase space volumes is described in Figure 1.1.

### b. Macrostates and Microstates

A *macrostate* of a system is said to be specified when the number of particles in each phase space volume  $dV$  is specified. That is, if the phase density is specified everywhere, then the macrostate of the system has been specified. Later we shall see how the phase density can be used to specify all the physical properties of the system.

To discuss the notion of a *microstate*, it must be assumed that there is a perceptible difference between particles, because in a microstate, in addition to the number of particles in each volume, it makes a difference *which* particles are in which volumes. If the specification of individual particles can be accomplished, then it can be said that a microstate has been specified. Clearly one macrostate could consist of many microstates. For example, the number of balls on a pool table might be said to be a macrostate, whereas the specification of which balls they are would denote a specific microstate. In a similar manner, the distribution of suits of playing cards in a bridge hand might be said to represent a macrostate, but the specific cards in each suit would specify the microstate.

### c. Probability and Statistical Equilibrium

If we were to create macrostates by assembling particles by randomly throwing them into various microstates, then the macrostate most likely to occur is the one with the greatest number of microstates. That is why a bridge hand consisting of 13 spades occurs so rarely compared to a hand with four spades and three hearts, three diamonds, or three clubs. If we consider a system where the particles are continually moving from one phase space volume to another, say, by collisions, then the most likely macrostate is the one with the largest number of associated microstates. There is an implicit assumption here that *all microstates are equally probable*. Is this reasonable?

Imagine a case where all the molecules in a room are gathered in one corner. This represents a particular microstate; a particularly unlikely one, we would think. Through random motions, it would take an extremely long time for the particles to return to that microstate. However, given the position and velocity of each particle in

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an ordinary room of gas, is this any more unlikely than each particle to returning to that specific position with the same velocity? The answer is no. Thus, if each microstate is equally probable, then the associated macrostates are not equally probable and it makes sense to search for the most probable macrostate of a system. In a system which is continually rearranging itself by collisions, the most probable macrostate becomes the most likely state in which to find the system. A system which is in its most probable macrostate is said to be in *statistical equilibrium*.

Many things can determine the most probable macrostate. Certainly the total number of particles allowed in each microstate and the total number of particles available to distribute will be important in determining the total number of microstates in a given macrostate. In addition, quantum mechanics places some conditions on our ability to distinguish particles and even limits how many of certain kinds of particles can be placed in a given volume of phase space. But, for the moment, let us put aside these considerations and concentrate on calculating the number of microstates in a particular macrostate.

4	3	2	1	0	Volume 1
0	1	2	3	4	Volume 2

**Figure 1.2** Shows a phase space composed of only two cells in which four particles reside. All possible macrostates are illustrated.

Consider a simple system consisting of only two phase space volumes and four particles (see Figure 1.2). There are precisely five different ways that the four particles can be arranged in the two volumes. Thus there are five macrostates of the system. But which is the most probable? Consider the second macrostate in Figure 1.2 (that is,  $N_1 = 3, N_2 = 1$ ). Here we have three particles in one volume and one particle in the other volume. If we regard the four particles as individuals, then there are four different ways in which we can place those four particles in the two volumes so that one volume has three and the other volume has only one (see Figure 1.3). Since the order in which the particles are placed in the volume does not matter, all permutations of the particles in any volume must be viewed as constituting the same microstate.

Now if we consider the total number of particles  $N$  to be arranged sequentially among  $m$  volumes, then the total number of sequences is simply  $N!$ . However, within each volume (say, the  $i$ th volume),  $N_i$  particles yield  $N_i!$  indistinguishable sequences which must be removed when the allowed number of microstates is counted. Thus the total number of allowed microstates in a given macrostate is

$$W = \frac{N!}{\prod_{i=1}^m N_i!} \tag{1.1.3}$$

<i>abc</i>	<i>abd</i>	<i>acd</i>	<i>bcd</i>	Volume 1
<i>d</i>	<i>c</i>	<i>b</i>	<i>a</i>	Volume 2

**Figure 1.3** Consider one of the macrostates in figure 1.2, specifically the state where  $N_1 = 3$ , and  $N_2 = 1$ . All the allowed microstates for distinguishable particles are shown.

For the five macrostates shown in Figure 1.2, the number of possible microstates is

$$\left. \begin{aligned} W_{4,0} &= 4!/4!0! = 1 \\ W_{3,1} &= 4!/3!1! = 4 \\ W_{2,2} &= 4!/2!2! = 6 \\ W_{1,3} &= 4!/1!3! = 4 \\ W_{0,4} &= 4!/0!4! = 1 \end{aligned} \right\} \tag{1.1.4}$$

Clearly  $W_{2,2}$  is the most probable macrostate of the five. The particle distribution of the most probable macro state is unique and is known as the *equilibrium macrostate*.

In a physical system where particle interactions are restricted to those between particles which make up the system, the number of microstates within the system changes after each interaction and, in general, increases, so that the macrostate of the system tends toward that with the largest number of microstates - the equilibrium macrostate. In this argument we assume that the interactions are uncorrelated and random. Under these conditions, a system which has reached its equilibrium macrostate is said to be in *strict thermodynamic equilibrium*. Note that interactions among particles which are not in strict thermodynamic equilibrium will tend to drive the system away from strict thermodynamic equilibrium and toward a different statistical equilibrium distribution. This is the case for stars near their surfaces.

The statistical distribution of microstates versus macrostates given by equation (1.1.3) is known as *Maxwell-Boltzmann statistics* and it gives excellent results for a classical gas in which the particles can be regarded as distinguishable. In

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a classical world, the position and momentum of a particle are sufficient to make it distinguishable from all other particles. However, the quantum mechanical picture of the physical world is quite different. So far, we have neglected both the Heisenberg uncertainty principle and the Pauli Exclusion Principle.

**d. Quantum Statistics**

Within the realm of classical physics, a particle occupies a point in phase space, and in some sense all particles are distinguishable by their positions and velocities. The phase space volumes are indeed differential and arbitrarily small. However, in the quantum mechanical view of the physical world, there is a limit to how well the position and momentum (velocity, if the mass is known) of any particle can be determined. Within that phase space volume, particles are indistinguishable. This limit is known as the *Heisenberg uncertainty principle* and it is stated as follows:

$$\Delta p \Delta x \geq h/2\pi \equiv \hbar \quad (1.1.5)$$

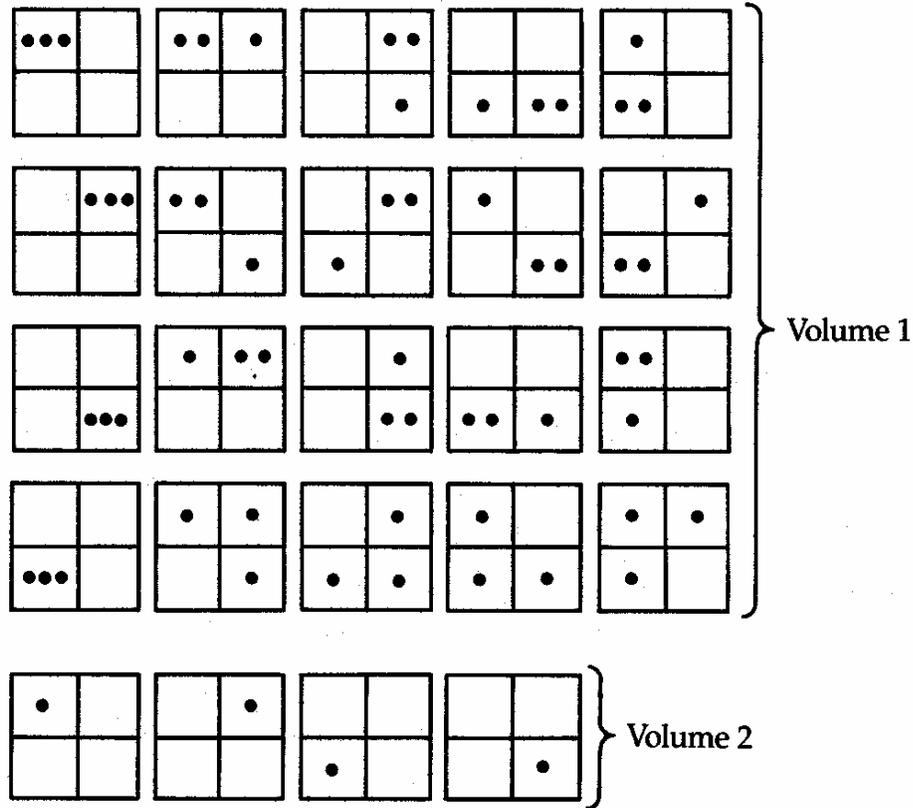
Thus the minimum phase space volume which quantum mechanics allows is of the order of  $h^3$ . To return to our analogy with Maxwell-Boltzmann statistics, let us subdivide the differential cell volumes into compartments of size  $h^3$  so that the total number of compartments is

$$n = dx_1 dx_2 dx_3 dp_1 dp_2 dp_3 / h^3 \quad (1.1.6)$$

Let us redraw the example in Figure 1.3 so that each cell in phase space is subdivided into four compartments within which the particles are indistinguishable. Figure 1.4 shows the arrangement for the four particles for the  $W_{3,1}$  macrostate for which there were only four allowed microstates under Maxwell-Boltzmann Statistics. Since the particles are now distinguishable within a cell, there are 20 separate ways to arrange the three particles in volume 1 and 4 ways to arrange the single particle in volume 2. The total number of allowed microstates for  $W_{3,1}$  is  $20 \times 4$ , or 80. Under these conditions the total number of microstates per macrostate is

$$W = \prod_i W_i \quad (1.1.7)$$

where  $W_i$  is the number of microstates per cell of phase space, which can be expressed in terms of the number of particles  $N_i$  in that cell.



**Figure 1.4** The same macrostate as figure 1.3 only now the cells of phase space are subdivided into four compartments within which particles are indistinguishable. All of the possible microstates are shown for the four particles.

Let us assume that there are  $n$  compartments in the  $i$ th cell which contains  $N_i$  particles. Now we have to arrange a sequence of  $n + N_i$  objects, since we have to consider both the particles and the compartments into which they can be placed. However, not all sequences are possible since we must always start a sequence with a compartment. After all we have to put the particle somewhere! Thus there are  $n$  sequences with  $N_i + n - 1$  items to be further arranged. So there are  $n[N_i + n - 1]!$  different ways to arrange the particles and compartments. We must eliminate all the permutations of the compartments because they reside within a cell and therefore represent the same microstate. But there are just  $n!$  of these. Similarly, the order in which the particles are added to the cell volume is just as irrelevant to the final microstate as it was under Maxwell-Boltzmann statistics. And so we must eliminate all the permutations of the  $N_i$  particles, which is just  $N_i!$ . Thus the number of microstates allowed for a given macrostate becomes

$$W_{B-E} = \prod_i n(N_i+n-1)! / N_i!n! = \prod_i (N_i+n-1)! / N_i!(n-1)! \quad (1.1.8)$$

The subscript "B-E" on W indicates that these statistics are known as *Bose-Einstein statistics* which allow for the Heisenberg uncertainty principle and the associated limit on the distinguishability of phase space volumes. We have assumed that an unlimited number of particles can be placed within a volume  $h^3$  of phase space, and those particles for which this is true are called *bosons*. Perhaps the most important representatives of the class of particles for stellar astrophysics are the photons. Thus, we may expect the statistical equilibrium distribution for photons to be different from that of classical particles described by Maxwell-Boltzmann statistics.

Within the domain of quantum mechanics, there are further constraints to consider. Most particles such as electrons and protons obey the Pauli Exclusion Principle, which basically says that there is a limit to the number of these particles that can be placed within a compartment of size  $h^3$ . Specifically, only one particle with a given set of quantum numbers may be placed in such a volume. However, two electrons which have their spins arranged in opposite directions but are otherwise identical can fit within a volume  $h^3$  of phase space. Since we can put no more than two of these particles in a compartment, let us consider phase space to be made up of  $2n$  half-compartments which are either full or empty. We could say that there are no more than  $2n$  things to be arranged in sequence and therefore no more than  $2n!$  allowed microstates. But, since each particle has to go somewhere, the number of filled compartments which have  $N_i!$  indistinguishable permutations are just  $N_i!$ . Similarly, the number of indistinguishable permutations of the empty compartments is  $(2n - N_i)!$ . Taking the product of all the allowed microstates for a given macrostate, we find that the total number of allowed microstates is

$$W_{F-D} = \prod_i (2n)! / N_i!(2n-N_i)! \quad (1.1.9)$$

The subscript "F-D" here refers to Enrico Fermi and P.A.M. Dirac who were responsible for the development of these statistics. These are the statistics we can expect to be followed by an electron gas and all other particles that obey the Pauli Exclusion Principle. Such particles are normally called *fermions*.

### e. Statistical Equilibrium for a Gas

To find the macrostate which represents a steady equilibrium for a gas, we follow basically the same procedures regardless of the statistics of the gas. In general, we wish to find that macrostate for which the number of microstates is a maximum. So by varying the number of particles in a cell volume we will search for

$dW = 0$ . Since  $\ln W$  is a monotonic function of  $W$ , any maximum of  $\ln W$  is a maximum of  $W$ . Thus we use the logarithm of equations (1.1.7) through (1.1.9) to search for the most probable macrostate of the distribution functions. These are

$$\left. \begin{aligned} \ln W_{M-B} &= \ln N! - \sum_i \ln(N_i!) \\ \ln W_{B-E} &= \sum_i \ln(n + N_i - 1)! - \ln N_i! - \ln(n - 1)! \\ \ln W_{F-D} &= \sum_i \ln(2n)! - \ln(2n - N_i)! - \ln N_i! \end{aligned} \right\} \quad (1.1.10)$$

The use of logarithms also makes it easier to deal with the factorials through the use of Stirling's formula for the logarithm of a factorial of a large number.

$$\ln N! \approx N \ln N - N \quad (1.1.11)$$

For a given volume of gas,  $dN = dn = 0$ . The variations of equations (1.1.10) become

$$\begin{aligned} \delta \ln W_{M-B} &= \sum_i \ln N_i dN_i = 0 \\ \delta \ln W_{B-E} &= \sum_i \ln[(n+N_i)/N_i] dN_i = 0 \\ \delta \ln W_{F-D} &= \sum_i \ln[(2n-N_i)/N_i] dN_i = 0 \end{aligned} \quad (1.1.12)$$

These are the equations of condition for the most probable macrostate for the three statistics which must be solved for the particle distribution  $N_i$ . We have additional constraints, which arise from the conservation of the particle number and energy, on the system which have not been directly incorporated into the equations of condition. These can be stated as follows:

$$\delta \left[ \sum_i N_i \right] = \delta N = 0, \quad \delta \left[ \sum_i w_i N_i \right] = \sum_i w_i \delta N_i = 0, \quad (1.1.13)$$

where  $w_i$  is the energy of an individual particle. Since these additional constraints represent new information about the system, we must find a way to incorporate them into the equations of condition. A standard method for doing this is known as the *method of Lagrange multipliers*. Since equations (1.1.13) represent quantities which are zero we can multiply them by arbitrary constants and add them to equations (1.1.12) to get

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$$\begin{aligned}
 \text{M-B:} \quad & \sum_i [\ln N_i - \ln \alpha_1 + \beta_1 w_i] \delta N_i = 0 \\
 \text{B-E:} \quad & \sum_i \{ \ln [ (n + N_i) / N_i ] - \ln \alpha_2 - \beta_2 w_i \} \delta N_i = 0 \quad (1.1.14) \\
 \text{F-D:} \quad & \sum_i \{ \ln [ (2n - N_i) / N_i ] - \ln \alpha_3 - \beta_3 w_i \} \delta N_i = 0
 \end{aligned}$$

Each term in equations (1.1.14) must be zero since the variations in  $N_i$  are arbitrary and any such variation must lead to the stationary, most probable, macrostate.

Thus,

$$\begin{aligned}
 \text{M-B:} \quad & N_i / \alpha_1 = \exp(-w_i \beta_1) \\
 \text{B-E:} \quad & n / N_i = \alpha_2 \exp(w_i \beta_2) - 1 \quad (1.1.15) \\
 \text{F-D:} \quad & 2n / N_i = \alpha_3 \exp(w_i \beta_3) + 1
 \end{aligned}$$

All that remains is to develop a physical interpretation of the undetermined parameters  $\alpha_j$  and  $\beta_j$ . Let us look at Maxwell-Boltzmann statistics for an example of how this is done. Since we have not said what  $\beta_1$  is, let us call it  $1/(kT)$ . Then

$$N_i = \alpha_1 e^{-w_i / (kT)} \quad (1.1.16)$$

If the cell volumes of phase space are not all the same size, it may be necessary to weight the number of particles to adjust for the different cell volumes. We call these weight functions  $g_i$ . The

$$N = \sum_i g_i N_i = \alpha_1 \sum_i g_i e^{-w_i / (kT)} \equiv \alpha_1 U(T) \quad (1.1.17)$$

The parameter  $U(T)$  is called the *partition function* and it depends on the composition of the gas and the parameter  $T$  alone. Now if the total energy of the gas is  $E$ , then

$$E = \sum_i g_i w_i N_i = \sum_i w_i g_i \alpha_1 e^{-w_i / kT} = \frac{[\sum_i w_i g_i N e^{-w_i / kT}]}{U(T)} = NkT \left[ \frac{d \ln U}{d \ln T} \right] \quad (1.1.18)$$

For a free particle like that found in a monatomic gas, the partition function<sup>1</sup> is (see also section 11.1b)

$$U(T) = \frac{(2\pi mkT)^{3/2}}{h^3} V \quad , \quad (1.1.19)$$

where  $V$  is the specific volume of the gas,  $m$  is the mass of the particle, and  $T$  is the kinetic temperature. Replacing  $d\ln U/d\ln T$  in equation (1.1.18) by its value obtained from equation (1.1.19), we get the familiar relation

$$E = \frac{3}{2} NkT \quad , \quad (1.1.20)$$

which is only correct if  $T$  is the kinetic temperature. Thus we arrive at a self-consistent solution if the parameter  $T$  is to be identified with the kinetic temperature.

The situation for a photon gas in the presence of material matter is somewhat simpler because the matter acts as a source and sinks for photons. Now we can no longer apply the constraint  $dN = 0$ . This is equivalent to adding  $\ln \alpha_2 = 0$  (i.e.,  $\alpha_2 = 1$ ) to the equations of condition. If we let  $\beta_2 = 1/(kT)$  as we did with the Maxwell-Boltzmann statistics, then the appropriate solution to the Bose-Einstein formula [equation (1.1.15)] becomes

$$\frac{N_i}{n} = \frac{1}{e^{hv/(kT)} - 1} \quad , \quad (1.1.21)$$

where the photon energy  $w_i$  has been replaced by  $hv$ . Since two photons in a volume  $h^3$  can be distinguished by their state of polarization, the number of phase space compartments is

$$n = (2/h^3) dx_1 dx_2 dx_3 dp_1 dp_2 dp_3 \quad (1.1.22)$$

We can replace the rectangular form of the momentum volume  $dp_1 dp_2 dp_3$ , by its spherical counterpart  $4\pi p^2 dp$  and remembering that the momentum of a photon is  $hv/c$ , we get

$$\frac{dN}{V} = \frac{8\pi v^2}{c^3} \frac{1}{e^{hv/(kT)} - 1} dv \quad . \quad (1.1.23)$$

Here we have replaced  $N_i$  with  $dN$ . This assumes that the number of particles in any phase space volume is small compared to the total number of particles. Since the energy per unit volume  $dE_v$  is just  $hv dN/V$ , we get the relation known as Planck's law or sometimes as the *blackbody law*:

$$dE_v = \frac{8\pi h}{c^3} \frac{v^3}{e^{h\nu/(kT)} - 1} dv \equiv \frac{4\pi}{c} B_v(T) \quad (1.1.24)$$

The parameter  $B_v(T)$  is known as the *Planck function*. This, then, is the distribution law for photons which are in strict thermodynamic equilibrium. If we were to consider the Bose-Einstein result for particles and let the number of Heisenberg compartments be much larger than the number of particles in any volume, we would recover the result for Maxwell-Boltzmann statistics. This is further justification for using the Maxwell-Boltzmann result for ordinary gases.

### f. Thermodynamic Equilibrium - Strict and Local

Let us now consider a two-component gas made up of material particles and photons. In stars, as throughout the universe, photons outnumber material particles by a large margin and continually undergo interactions with matter. Indeed, it is the interplay between the photon gas and the matter which is the primary subject of this book. If both components of the gas are in statistical equilibrium, then we should expect the distribution of the photons to be given by Planck's law and the distribution of particle energies to be given by the Maxwell-Boltzmann statistics. In some cases, when the density of matter becomes very high and the various cells of phase space become filled, it may be necessary to use Fermi-Dirac statistics to describe some aspects of the matter. When both the photon and the material matter components of the gas are in statistical equilibrium with each other, we say that the gas is in strict thermodynamic equilibrium. If, for whatever reason, the photons depart from their statistical equilibrium (i.e., from Planck's law), but the material matter continues to follow Maxwell-Boltzmann Statistics (i.e., to behave as if it were still in thermodynamic equilibrium), we say that the gas (material component) is in local thermodynamic equilibrium (LTE).

## 1.2 Transport Phenomena

### a. Boltzmann Transport Equation

It is one thing to describe the behavior of matter and photons in equilibrium, but stars shine. Therefore energy must flow from the interior to the surface regions of the star and the details of the flow play a dominant role in determining the resultant structure and evolution of the star. We now turn to an extremely simple description of how this flow can be quantified; this treatment is due to Ludwig Boltzmann and should not be confused with the Boltzmann formula of Maxwell-Boltzmann statistics. Although the ideas of Boltzmann are conceptually

simple, many of the most fundamental equations of theoretical physics are obtained from them.

Basically the Boltzmann transport equation arises from considering what can happen to a collection of particles as they flow through a volume of phase space. Our prototypical volume of phase space was a six-dimensional "cube", which implies that it has five-dimensional "faces". The Boltzmann transport equation basically expresses the change in the phase density within a differential volume, in terms of the flow through these faces, and the creation or destruction of particles within that volume.

For the moment, let us call the six coordinates of this space  $x_i$  remembering that the first three refer to the spatial coordinates and the last three refer to the momentum coordinates. If the "area" of one of the five-dimensional "faces" is  $A$ , the particle density is  $N/V$ , and the flow velocity is  $v$ , then the inflow of particles across that face in time  $dt$  is

$$v \frac{N}{V} A dt = \left[ \frac{dx_i}{dt} f(x_i, t) \frac{dV}{dx_i} \right] dt \quad (1.2.1)$$

Similarly, the number of particles flowing out of the opposite face located  $dx_i$  away is

$$\left[ \frac{dx_i}{dt} f(x_i + dx_i, t) \frac{dV}{dx_i} \right] dt = \text{number of outflowing particles} \quad (1.2.2)$$

The net change due to flow in and out of the six-dimensional volume is obtained by calculating the difference between the inflow and outflow and summing over all faces of the volume:

$$\sum_i \frac{dx_i}{dt} \left[ \frac{f(x_i + dx_i, t) - f(x_i, t)}{dx_i} \right] dV dt = \sum_i \frac{dx_i}{dt} \frac{\partial f}{\partial x_i} dV dt \quad (1.2.3)$$

Note that the sense of equation (1.2.3) is such that if the inflow exceeds the outflow, the net flow is considered negative. Now this flow change must be equal to the negative time rate of change of the phase density (i.e.,  $df/dt$ ). We can split the total time rate of change of the phase density into that part which represents changes due to the differential flow  $\langle f \rangle / \langle t \rangle$  and that part which we call the creation rate  $S$ . Equating the flow divergence with the local temporal change in the phase density, we have

$$\sum_i \frac{dx_i}{dt} \frac{\partial f}{\partial x_i} dV dt = - \left( \frac{\partial f}{\partial t} - S \right) dV dt \quad (1.2.4)$$

Rewriting our phase space coordinates  $x_i$  in terms of the spatial and momentum coordinates and using the old notation of Isaac Newton to denote total differentiation with respect to time (i.e., the dot  $\dot{\phantom{x}}$ ) we get

$$\frac{\partial f(x_i, p_i, t)}{\partial t} + \sum_{i=1}^3 \left( \dot{x}_i \frac{\partial f}{\partial x_i} + \dot{p}_i \frac{\partial f}{\partial p_i} \right) = S \quad (1.2.5)$$

This is known as the Boltzmann transport equation and can be written in several different ways. In vector notation we get

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f - \frac{1}{m} \nabla \Phi \cdot \nabla_v f = S \quad (1.2.6)$$

Here the potential gradient  $\nabla \Phi$  has replaced the momentum time derivative while  $\nabla_v$  is a gradient with respect to velocity. The quantity  $m$  is the mass of a typical particle. It is also not unusual to find the Boltzmann transport equation written in terms of the total Stokes time derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \quad (1.2.7)$$

If we take  $\vec{v}$  to be a six dimensional "velocity" and  $\nabla$  to be a six- dimensional gradient, then the Boltzmann transport equation takes the form

$$\frac{Df}{Dt} = S \quad (1.2.8)$$

Although this form of the Boltzmann transport equation is extremely general, much can be learned from the solution of the homogeneous equation. This implies that  $S = 0$  and that no particles are created or destroyed in phase space.

### b. Homogeneous Boltzmann Transport Equation and Liouville's Theorem

Remember that the right-hand side of the Boltzmann transport equation is a measure of the rate at which particles are created or destroyed in the phase space volume. Note that creation or destruction in phase space includes a good deal more than the conventional spatial creation or destruction of particles. To be sure, that type of change is included, but in addition processes which change a particle's position in momentum space may move a particle in or out of such a volume. The detailed nature of such processes will interest us later, but for the moment let us consider a common and important form of the Boltzmann transport equation, namely that where the right-hand side is zero. This is known as the

*homogeneous Boltzmann transport equation*. It is also better known as *Liouville's theorem of statistical mechanics*. In the literature of stellar dynamics, it is also occasionally referred to as *Jeans' theorem*<sup>2</sup> for Sir James Jeans was the first to explore its implications for stellar systems. By setting the right-hand side of the Boltzmann transport equation to zero, we have removed the effects of collisions from the system, with the result that the density of points in phase space is constant. Liouville's theorem is usually generalized to include sets or ensembles of particles. For this generalization phase space is expanded to  $6N$  dimensions, so that each particle of an ensemble has six position and momentum coordinates which are linearly independent of the coordinates of every other particle. This space is often called *configuration space*, since the entire ensemble of particles is represented by a point and its temporal history by a curve in this  $6N$ -dimensional space. Liouville's theorem holds here and implies that the density of points (ensembles) in configuration space is constant. This, in turn, can be used to demonstrate the determinism and uniqueness of Newtonian mechanics. If the configuration density is constant, it is impossible for two ensemble paths to cross, for to do so, one path would have to cross a volume element surrounding a point on the other path, thereby changing the density. If no two paths can cross, then it is impossible for any two ensembles to ever have exactly the same values of position and momentum for all their particles. Equivalently, the initial conditions of an ensemble of particles uniquely specify its path in configuration space. This is not offered as a rigorous proof, only as a plausibility argument. More rigorous proofs can be found in most good books on classical mechanics<sup>3,4</sup>. Since Liouville's theorem deals with configuration space, it is sometimes considered more fundamental than the Boltzmann transport equation; but for our purposes the expression containing the creation rate  $S$  will be required and therefore will prove more useful.

### c. Moments of the Boltzmann Transport Equation and Conservation Laws

By the *moment* of a function we mean the integral of some property of interest, weighted by its distribution function, over the space for which the distribution function is defined. Common examples of such moments can be found in statistics. The mean of a distribution function is simply the first moment of the distribution function, and the variance can be simply related to the second moment. In general, if the distribution function is analytic, all the information contained in the function is also contained in the moments of that function.

The complete solution to the Boltzmann transport equation is, in general, extremely difficult and usually would contain much more information about the system than we wish to know. The process of integrating the function over its defined space to obtain a specific moment removes or averages out much of the

information about the function. However, this process also usually yields equations which are much easier to solve. Thus we trade off information for the ability to solve the resulting equations, and we obtain some explicit properties of the system of interest. This is a standard "trick" of mathematical physics and one which is employed over and over throughout this book. Almost every instance of this type carries with it the name of some distinguished scientist or is identified with some fundamental conservation law, but the process of its formulation and its origin are basically the same.

We define the *n*th moment of a function *f* as

$$M_n[f(x)] = \int x^n f(x) dx. \quad (1.2.9)$$

By multiplying the Boltzmann equation by powers of the position and velocity and integrating over the appropriate dimensions of phase space, we can generate equations relating the various moments of the phase density  $f(\bar{x}, \bar{v})$ . In general, such a process always generates two moments of different order *n*, so that a succession of moment taking always generates one more moment than is contained in the resulting equations. Some additional property of the system will have to be invoked to relate the last generated higher moment to one of lower order, in order to close the system of equations and allow for a solution. To demonstrate this process, we show how the equation of continuity, the Euler-Lagrange equations of hydrodynamic flow, and the virial theorem can all be obtained from moments of the Boltzmann transport equation.

**Continuity Equation and the Zeroth Velocity Moment** Although most moments, particularly in statistics, are normalized by the integral of the distribution function itself, we have chosen not to do so here because the integral of the phase density *f* over all velocity space has a particularly important physical meaning, namely, the local spatial density.

$$\rho = m \int_{-\infty}^{+\infty} f(\bar{x}, \bar{v}) d\bar{v} \quad (1.2.10)$$

By  $d\bar{v}$  we mean that the integration is to be carried out over all three velocity coordinates  $v_1$ ,  $v_2$ , and  $v_3$ . A pedant might correctly observe that the velocity integrals should only range from  $-c$  to  $+c$ , but for our purposes the Newtonian view will suffice. Integration over momentum space will properly preserve the limits. Now let us integrate the component form of equation (1.2.5) over all velocity space to generate an equation for the local density. Thus,

$$\int_{-\infty}^{+\infty} \left( \frac{\partial f}{\partial t} + \sum_{i=1}^3 v_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^3 \dot{v}_i \frac{\partial f}{\partial v_i} \right) d\vec{v} = \int_{-\infty}^{+\infty} S d\vec{v} \quad (1.2.11)$$

Since the velocity and space coordinates are linearly independent, all time and space operators are independent of the velocity integrals. The integral of the creation rate  $S$  over all velocity space becomes simply the creation rate for particles in physical space, which we call  $\mathfrak{S}$ . By noting that the two summations in equation (1.2.11) are essentially scalar products, we can rewrite that moment and get

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} f d\vec{v} + \int_{-\infty}^{+\infty} \vec{v} \cdot \nabla f d\vec{v} + \int_{-\infty}^{+\infty} \dot{\vec{v}} \cdot \nabla_v f d\vec{v} = \mathfrak{S} \quad (1.2.12)$$

It is clear from the definition of  $\rho$  that the first term is the partial derivative of the local particle density. The second term can be modified by use of the vector identity

$$\vec{v} \cdot \nabla f = \nabla \cdot (f\vec{v}) - f \nabla \cdot \vec{v} \quad (1.2.13)$$

and by noting that  $\nabla \cdot \vec{v} = 0$ , since space and velocity coordinates are independent. If the particles move in response, to a central force, then we may relate their accelerations  $\dot{\vec{v}}$  to the gradient of a potential which depends on only position and not velocity. The last term then takes the form  $(\nabla \cdot \Phi / m) \cdot \int \nabla_v f(\vec{v}) d\vec{v}$ . If we further require that  $f(\vec{v})$  be bounded (i.e., that there be no particles with infinite velocity), then since the integral and gradient operators basically undo each other, the integral and hence the last term of equation (1.2.12) vanish, leaving

$$\frac{\partial n}{\partial t} + \nabla \cdot \int_{-\infty}^{+\infty} \vec{v} f d\vec{v} = \mathfrak{S} \quad (1.2.14)$$

The second term in equation (1.2.14) is the first velocity moment of the phase density and illustrates the manner by which higher moments are always generated by the moment analysis of the Boltzmann transport equation. However, the physical interpretation of this moment is clear. Except for a normalization scalar, the second term is a measure of the mean flow rate of the material. Thus, we can define a mean flow velocity  $\vec{u}$

$$\vec{u} = \frac{\int \vec{v} f(\vec{v}) d\vec{v}}{\int f(\vec{v}) d\vec{v}} \quad (1.2.15)$$

which, upon multiplying by the particle mass, enables us to obtain the familiar form of the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = \mathfrak{S} m \quad (1.2.16)$$

This equation basically says that the explicit time variation of the density plus density changes resulting from the divergence of the flow is equal to the local creation or destruction of material  $\mathfrak{S}$ .

***Euler-Lagrange Equations of Hydrodynamic Flow and the First Velocity Moment of the Boltzmann Transport Equation*** The zeroth moment of the transport equation provided insight into the way in which matter is conserved in a flowing medium. Multiplying the Boltzmann transport equation by the velocity and integrating over all velocity space will produce momentum-like moments, and so we might expect that such operations will also produce an expression of the conservation of momentum. This is indeed the case. However, keep in mind that the velocity is a vector quantity, and so the moment analysis will produce a vector equation rather than the scalar equation, as was the case with the equation of continuity. Multiplying the Boltzmann transport equation by the local particle velocity  $\vec{v}$ , we get

$$\int_{-\infty}^{+\infty} \vec{v} \frac{\partial f}{\partial t} d\vec{v} + \int_{-\infty}^{+\infty} \vec{v}(\vec{v} \cdot \nabla f) d\vec{v} + \int_{-\infty}^{+\infty} \vec{v}(\vec{v} \cdot \nabla_v f) d\vec{v} = \int_{-\infty}^{+\infty} \vec{v} S d\vec{v} \quad (1.2.17)$$

We can make use of most of the tricks that were used in the derivation of the continuity equation (1.2.16). The first term can be expressed in terms of the mean flow velocity [equation (1.2.15)] while the second term can be expressed as

$$\int_{-\infty}^{+\infty} \vec{v} \nabla \cdot (\vec{v} f) d\vec{v} \quad (1.2.18)$$

by using the vector identity given by equation (1.2.13). Since the quantity in parentheses of the third term in equation (1.2.17) is a scalar and since the particle accelerations depend on position only, we can move them and the vector scalar product outside the velocity integral and re-express them in terms of a potential, so the third term becomes

$$\frac{1}{m} \nabla \Phi \cdot \int_{-\infty}^{+\infty} (\nabla_v f) \vec{v} d\vec{v} \quad (1.2.19)$$

The integrand of equation (1.2.19) is not a simple scalar or vector, but is the vector outer, or tensor, product of the velocity gradient of  $f$  with the vector velocity  $\vec{v}$  itself. However, the vector identity given by equation (1.2.13) still applies if the scalar product is replaced with the vector outer product, so that the integrand in equation (1.2.19) becomes

$$(\nabla_v f) \vec{v} = \nabla_v (f \vec{v}) - f \nabla_v \vec{v} = \nabla_v (f \vec{v}) - \mathbf{1} f \quad (1.2.20)$$

The quantity  $\mathbf{1}$  is the unit tensor and has elements of the Kronecker delta  $\delta_{ij}$  whose

elements are zero when  $i \neq j$  and 1 when  $i = j$ . Again, as long as  $f$  is bounded, the integral over all velocity space involving the velocity gradient of  $f$  will be zero, and the first velocity moment of the Boltzmann transport equation becomes

$$\frac{\partial(n\bar{u})}{\partial t} + \int_{-\infty}^{+\infty} \bar{v} \nabla \cdot (\bar{v} f) d\bar{v} - n \nabla \Phi = \int_{-\infty}^{+\infty} S \bar{v} d\bar{v} \quad (1.2.21)$$

Differentiating the first term and using the continuity equation (1.2.14) to eliminate  $\partial n / \partial t$ , we get

$$\begin{aligned} n \frac{\partial \bar{u}}{\partial t} - (\bar{u} \cdot \nabla n + n \nabla \cdot \bar{u}) \bar{u} + \int_{-\infty}^{+\infty} \bar{v} \nabla \cdot (\bar{v} f) d\bar{v} + \rho \nabla \Phi \\ = \int_{-\infty}^{+\infty} \bar{v} S d\bar{v} - \int_{-\infty}^{+\infty} \bar{u} S d\bar{v} \end{aligned} \quad (1.2.22)$$

Since  $\nabla \cdot \bar{v}$  is zero and the velocity and space coordinates are independent, we may rewrite the third term in terms of a velocity tensor defined as

$$\mathbf{u} = \frac{\int_{-\infty}^{+\infty} \bar{v} \bar{v} f(v) d\bar{v}}{\int_{-\infty}^{+\infty} f(v) d\bar{v}} \quad (1.2.23)$$

Some rearranging and the use of a few vector identities lead to

$$\rho \frac{\partial \bar{u}}{\partial t} + \rho (\bar{u} \cdot \nabla) \bar{u} + \nabla \cdot [\rho (\mathbf{u} - \bar{u} \bar{u})] + \rho \nabla \Phi = \int_{-\infty}^{+\infty} m S (\bar{v} - \bar{u}) d\bar{v} \quad (1.2.24)$$

The quantity in brackets of the third term is sometimes called the *pressure tensor*. A density  $\rho$  times a velocity squared is an energy density, which has the units of pressure. We can rewrite that term so it has the form

$$\mathbf{P} = \frac{\rho \int_{-\infty}^{+\infty} f(v) (\bar{v} - \bar{u})(\bar{v} - \bar{u}) d\bar{v}}{\int_{-\infty}^{+\infty} f(v) d\bar{v}} \quad (1.2.25)$$

which shows the form of a moment of  $f$ . In this instance the moment is a tensor that more or less describes the difference between the local flow indicated by  $\bar{v}$  and the mean flow  $\bar{u}$ . The form of the moment is that of a variance, and the tensor, in general, consists of nine components. Each component measures the net momentum transfer (or contribution to the local energy density) across a surface associated with that coordinate which results from the net flow coming from another coordinate. Thus the third term is simply the divergence of the pressure tensor which is a vector quantity, and the first velocity moment of the Boltzmann transport equation becomes

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$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} = -\nabla \Phi - \frac{1}{\rho} \nabla \cdot \mathbf{P} + \frac{1}{\rho} \int_{-\infty}^{+\infty} mS(\bar{\mathbf{v}} - \bar{\mathbf{u}}) d\bar{\mathbf{v}} \quad (1.2.26)$$

This set of vector equations is known as the *Euler-Lagrange equations of hydrodynamic flow* and they are derived here in their most general form.

It is common to make some further assumptions concerning the flow to further simplify these flow equations. If we consider the common physical situation where there are many collisions in the gas, then there is a tendency to randomize the local velocity field  $\bar{\mathbf{v}}$  and thus to make  $f(\bar{\mathbf{v}}) = f(-\bar{\mathbf{v}})$ . Under these conditions, the pressure tensor becomes diagonal, all elements are equal, and its divergence can be written as the gradient of some scalar which we call the pressure  $\mathbf{P}$ . In addition, the creation rate  $\mathcal{S}$  in equation (1.2.26) which involves the effects of collisions will also become symmetric in velocity, which means that the entire integral over velocity space will vanish. This single assumption leads to the simpler and more familiar expression for hydrodynamic flow, namely

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla) \bar{\mathbf{u}} = -\nabla \Phi - \frac{\nabla P}{\rho} \quad (1.2.27)$$

This assumption is necessary to close the moment analysis in that it provides a relationship between the pressure tensor and the scalar pressure  $\mathbf{P}$ . From the definition of the pressure tensor, under the assumption of a nearly isotropic velocity field,  $\mathbf{P}$  will be  $\mathbf{P}(\rho)$  and an expression known as an *equation of state* will exist. It is this additional equation that will complete the closure of the hydrodynamic flow equations and will allow for solutions. It is also worth remembering that if the mean flow velocity  $\bar{\mathbf{u}}$  is very large compared to the velocities produced by collisions, then the above assumption is invalid, no scalar equation of state will exist, and the full-blown equations of hydrodynamic flow given by equation (1.2.26) must be solved. In addition, a good deal of additional information about the system must be known so that a tensor equation of state can be found and the creation term can be evaluated.

It is worth making one further assumption regarding the flow equations. Consider the case where the flow is zero and the material is quiescent. The entire left-hand side of equation (1.2.27) is now zero, and the assumption of an isotropic velocity field produced by random collisions holds exactly. The Euler-Lagrange equations of hydrodynamic flow now take the particularly simple form

$$\nabla P = -\rho \nabla \Phi \quad (1.2.28)$$

which is known as the *equation of hydrostatic equilibrium*. This equation is usually cited as an expression of the conservation of linear momentum. Thus the zeroth moment of the Boltzmann transport equation results in the conservation of matter, whereas the first velocity moment yields equations which represent the conservation of linear momentum. You should not be surprised that the second velocity moment will produce an expression for the conservation of energy. So far we have considered

moment analysis involving velocity space alone. Later we shall see how moments taken over some dimensions of physical space can produce the diffusion approximation so important to the transfer of photons. As you might expect, moments taken over all physical space should yield "conservation laws" which apply to an entire system. There is one such example worth considering.

**Boltzmann Transport Equation and the Virial Theorem** The Virial theorem of classical mechanics has a long and venerable history which begins with the early work of Joseph Lagrange and Karl Jacobi. However, the theorem takes its name from work of Rudolf Clausius in the early phases of what we now call thermodynamics. Its most general expression and its relation to both subjects can be nicely seen by obtaining the virial theorem from the Boltzmann transport equation. Let us start with the Euler-Lagrange equations of hydrodynamic flow, which already represent the first velocity moment of the transport equation. These are vector equations, and so we may obtain a scalar result by taking the scalar product of a position vector with the flow equations and integrating over all space which contains the system. This effectively produces a second moment, albeit with mixed moments, of the transport equation. In the 1960s, S. Chandrasekhar and collaborators developed an entire series of Virial-like equations by taking the vector outer (or tensor) product of a position vector with the Euler-Lagrange flow equations. This operation produced a series of tensor equations which they employed for the study of stellar structure. Expressions which Chandrasekhar termed "higher-order virial equations" were obtained by taking additional moments in the spatial coordinate  $r$ . However, the use of higher moments makes the relationship to the Virial theorem somewhat obscure.

The origin of the position vector is important only in the interpretation of some of the terms which will arise in the expression. Remembering that the left-hand side of equation (1.2.27) is the total time derivative of the flow velocity  $\bar{u}$ , we see that this first spatial moment equation becomes

$$\int_V \rho \vec{r} \cdot \frac{d\bar{u}}{dt} dV + \int_V \rho \vec{r} \cdot \nabla \Phi dV + \int_V \vec{r} \cdot \nabla P dV = 0 \quad (1.2.29)$$

With some generality

$$\int_V \rho \frac{dQ}{dt} dV = \frac{d}{dt} \int_V \rho Q dV \quad (1.2.30)$$

Since  $\bar{u}$  is just the time rate of change of position, we can rewrite  $\vec{r} \cdot (d\bar{u}/dt)$  so that the first integral of equation (1.2.29) becomes

$$\int_V \rho \ddot{\mathbf{r}} \cdot \frac{d\mathbf{u}}{dt} dV = \frac{1}{2} \frac{d^2}{dt^2} \int_V \rho r^2 dV - 2\mathbf{T} \quad (1.2.31)$$

Here  $\mathbf{T}$  is just the total kinetic energy due to the mass motions, as described by  $\bar{\mathbf{u}}$ , of the system, and the integral can be interpreted as the moment of inertia about the *center*, or origin, of the coordinate frame which defines  $\bar{\mathbf{r}}$ . The third integral in equation (1.2.29) can be rewritten by using the product law of differentiation and the divergence theorem:

$$\int_V \ddot{\mathbf{r}} \cdot \nabla P dV = \int_V \nabla \cdot (\ddot{\mathbf{r}} P) dV - \int_V P \nabla \cdot \ddot{\mathbf{r}} dV = \oint_S P \ddot{\mathbf{r}} \cdot \hat{\mathbf{n}} dA - 3 \int_V P dV \quad (1.2.32)$$

It is also worth noting that  $\nabla \cdot \bar{\mathbf{r}} = 3$ . We usually take the volume enclosing the object to be sufficiently large that  $\mathbf{P}_s = 0$ . If we now make use of the ideal gas law [which we derive in the next section along with the fact that the internal kinetic energy density of an ideal gas is  $\varepsilon = \frac{3}{2} \rho kT / (\mu m_h)$ ], we can replace the pressure  $\mathbf{P}$  in the last integral of equation (1.2.32) with  $(2/3)\varepsilon$ . The integral then yields twice the total internal kinetic energy of the system, and our moment equation becomes

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 2(\mathbf{T} + \mathbf{U}) - \int_V \rho \ddot{\mathbf{r}} \cdot \nabla \Phi dV \quad (1.2.33)$$

Here  $I$  is the moment of inertia about the origin of the coordinate system, and  $\mathbf{U}$  is the total internal kinetic energy resulting from the random motion of the molecules of the gas. The last term on the right is known as the *Virial of Clausius* whence the theorem takes its name. The units of that term are force times distance, so it is also an energy-like term and can be expressed in terms of the total potential energy of the system. Indeed, if the force law governing the particles of the system behaves as  $1/r^2$ , the Virial of Clausius is just the total potential energy<sup>5</sup>. This leads to an expression sometimes called *Lagrange's identity* which was first developed in full generality by Karl Jacobi and is also called the *non-averaged* form of the Virial theorem

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 2(\mathbf{T} + \mathbf{U}) + \mathbf{\Omega} \quad (1.2.34)$$

If we consider a system in equilibrium or at least a long-term steady state, so that the time average of equation (1.2.34) removes the accelerative changes of the moment of inertia (i.e.,  $\langle d^2 I / dt^2 \rangle = 0$ ), then we get the more familiar form of the Virial theorem, namely,

$$2\langle \mathbf{T} \rangle + 2\langle \mathbf{U} \rangle + \langle \mathbf{\Omega} \rangle = 0 \quad (1.2.35)$$

It is worth mentioning that the use of the Virial theorem in astronomy often

replaces the time averages with ensemble averages over all available phase space. The theorem which permits this is known as the *Ergodic theorem*, and all of thermodynamics rests on it. Although such a replacement is legitimate for large systems consisting of many particles, such as a star, considerable care must be exercised in applying it to stellar or extra-galactic systems having only a few members. However, the Virial theorem itself has basically the form and origin of a conservation law, and when the conditions of the theorem's derivation hold, it must apply.

### 1.3 Equation of State for an Ideal Gas and Degenerate Matter

Formulation of the Boltzmann transport equation also provides an ideal setting for the formulation of the equation of state for a gas under wide-ranging conditions. The statistical distribution functions developed in Section 1.1 give us the distribution functions for particles which depend largely on how filled phase space happens to be. Those functions relate the particle energy and the kinetic temperature to the distribution of particles in phase space. This is exactly what is meant by  $f(\vec{v})$ . Thus we can calculate the expected relationship between the pressure as given by the pressure tensor and the state variables of the distribution function. The result is known as the *equation of state*.

As given in equation (1.2.24), the pressure tensor is  $\mathbf{p}(\bar{u} - \bar{u}\bar{u})$ . If  $f(\vec{v})$  is symmetric in  $\vec{v}$ , then  $\bar{u}$  must be zero (or there must exist an inertial coordinate system in which  $\bar{u}$  is zero), and the divergence of the pressure tensor can be replaced by the gradient of a scalar, which we call the *gas pressure*, and will be given by

$$P = \rho \frac{\int_{-\infty}^{+\infty} v^2 f(v) d\vec{v}}{\int_{-\infty}^{+\infty} f(v) d\vec{v}} \quad (1.3.1)$$

From Maxwell-Boltzmann statistics, the distribution function of particles, in terms of their velocity, was given by equation (1.1.16). If we regard the number of cells of phase space to be very large, we can replace  $N_i$  by  $dN$  and consider equation (1.1.16) to give the distribution function  $f(\vec{v})$ , so that

$$f(v) = \text{constant} \times e^{-mv^2/(2kT)} \quad (1.3.2)$$

Now, in general,

$$\int_0^{+\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n} \left(\frac{\pi}{a}\right)^{1/2} \quad (1.3.3)$$

Substitution of equation (1.3.2) into equation (1.3.1) therefore yields

$$P = \frac{\rho k T}{m} = nkT \quad (1.3.4)$$

This is known as the *ideal-gas law* and it is the appropriate equation of state for a gas obeying Maxwell-Boltzmann statistics. That is, we may confidently expect that this simple formula will provide the correct relation among  $P$ ,  $T$ , and  $\rho$  as long as the cells of phase space do not become overly filled and quantum effects become important. If the density is increased without a corresponding increase in particle energy, a point will come when the available cells of phase space begin to fill up in accordance with the Pauli Exclusion Principle. As the most "popular" cells in phase space become filled, the particles will have to spill over into adjacent cells, producing a distortion in the distribution function (see Figure 1.5). When this happens, the gas is said to become *partially degenerate*. Figure 1.5 shows this effect and indicates a way to quantify the effect. We define a momentum  $p_0$  as that momentum above which there are just enough particles to fill the remaining phase space cells below  $p_0$ . Thus

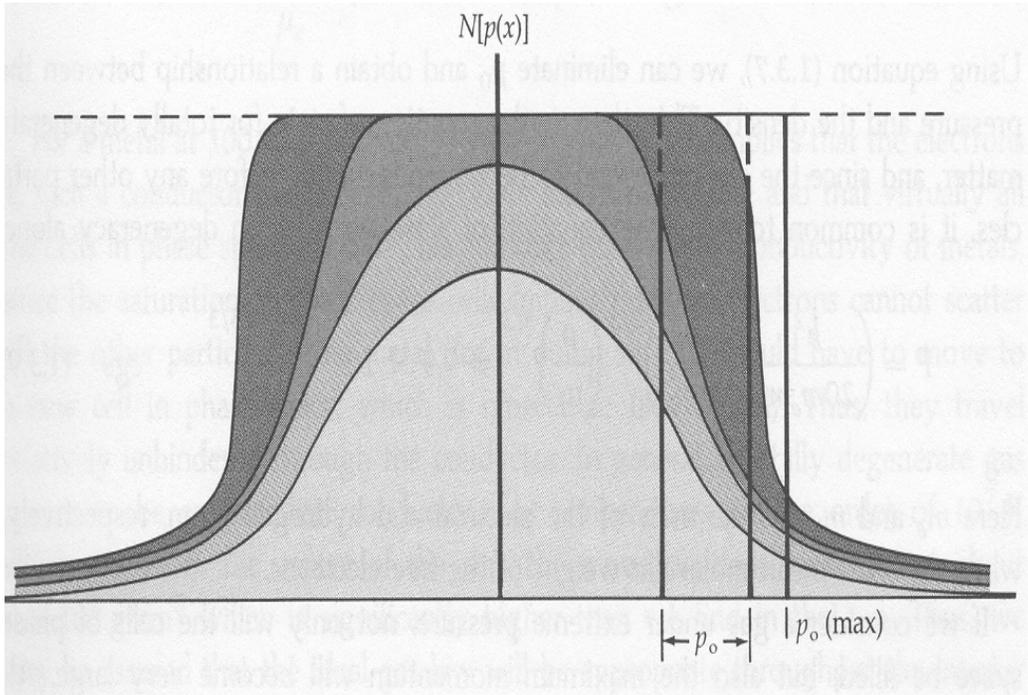
$$\int_0^{p_0} [N_{\max} - N(p)] dp = \int_{p_0}^{\infty} N(p) dp \quad (1.3.5)$$

If we make the approximation that all the spaces in phase space are filled (i.e., a negligible number of particles exist with momentum above  $p_0$ ), then the momentum distribution of the particles can be represented by a section of a sphere in momentum space so that

$$n(p) dp = 2 \left( 4\pi p^2 \frac{dp}{h^3} \right) \quad (1.3.6)$$

The factor of 2 arises because the electron can have two spin states in a cell of size  $h^3$ . The number density of particles can then be given in terms of  $p_0$  as

$$n = 2 \int_0^{p_0} \frac{4\pi p^2}{h^3} dp = \frac{8\pi p_0^3}{3h^3} \quad (1.3.7)$$



**Figure 1.5** Shows a momentum cross section of phase space with different particle densities. As the volume saturates, the distribution function departs further and further from Maxwellian. The Fermi momentum  $p_0$ , represents that momentum such that the particles having greater momentum would just fill the momentum states below it.

We have already developed a relation for the scalar pressure in terms of the velocity distribution under the assumption of an isotropic velocity field in equation (1.3.1), and we need only replace the velocity distribution  $f(\vec{v})$  with a distribution function of momentum. However, we must remember that the integral in equation (1.3.1) is actually three integrals over each velocity coordinate which will all have the same value for an isotropic velocity field. The three integrals corresponding to the three components of velocity in equation (1.3.1) are equal for spherical momentum space. Therefore one-third of the scalar form of equation (1.3.1) will represent the total contribution of the momentum to the pressure. Thus the pressure can be expressed in terms of the maximum momentum  $p_0$ , often known as the *Fermi momentum*, as

$$P = \frac{1}{3} \int_0^{p_0} \frac{p^2}{m} n(p) dp = \frac{1}{3} \int_0^{p_0} \frac{p^2}{m} \left( \frac{8\pi p^2}{h^3} \right) dp = \frac{8\pi p_0^5}{15mh^3} = \frac{h^2}{20m} \left( \frac{3}{\pi} \right)^{2/3} n^{5/3} \quad (1.3.8)$$

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Using equation (1.3.7), we can eliminate  $p_0$  and obtain a relationship between the pressure and the density. This, then, is the equation of state for totally degenerate

matter, and since the electrons tend to become degenerate before any other particles, it is common to write the equation of state for electron degeneracy alone.

$$P = \left( \frac{h^2}{20m_e m_h} \right) \left( \frac{3}{\pi m_h} \right)^{2/3} \left( \frac{\rho}{\mu_e} \right)^{5/3} = 9.9 \times 10^{12} \left( \frac{\rho}{\mu_e} \right)^{5/3} \quad \text{cgs} \quad (1.3.9)$$

Here  $m_e$  and  $m_h$  are the mass of the electron and hydrogen atom, respectively, while  $\mu_e$  is the mean molecular weight of the free electrons.

If we consider a gas under extreme pressure, not only will the cells of phase space be filled, but also the maximum momentum will become very large. Although the mass  $m$  and momentum  $p$  both approach infinity as the particle energy increases, their ratio  $p/m$  does not. It remains finite and approaches the speed of light  $c$ . Since these particles also make the largest contribution to the pressure, we can estimate the effect of having a relativistically degenerate gas by replacing  $p/m$  by  $c$  in equation (1.3.8), and we get

$$P = \frac{1}{3} \int_0^{p_0} \frac{p^2}{m} n(p) dp = \frac{1}{3} \int_0^{p_0} \frac{8\pi p^3 c}{h^3} dp = \frac{2\pi p_0^4 c}{3h^3} \quad (1.3.10)$$

which leads to an equation of state that depends on  $p^{4/3}$  instead of  $p^{5/3}$ , as in the case of nonrelativistic degeneracy. Eliminating  $p_0$ , we obtain for the electron degeneracy

$$P = (hc/8m_h)(3/\pi m_h)^{1/3} (p/m_e)^{4/3} = 1.231 \times 10^{15} (p/m_e)^{4/3} \quad (\text{cgs}) \quad (1.3.11)$$

The equations of state for degenerate matter that we have derived represent limiting conditions and are never exactly realized. In real situations we must consider the transition between the ideal-gas law and total degeneracy as well as the transition between nonrelativistic and completely relativistic degeneracy. One way to identify the range of state variables for which we can expect a transition zone is to equate the various equations of state and to solve for the range of state variables involved. Equating the ideal-gas law [equation (1.3.4)] with the equation for a totally degenerate gas [equation (1.3.9)], we can determine the range of density  $\rho_t$  and temperature  $T_t$  which lie in the transition zone between the two equations of state

$$\frac{\rho_t}{\mu_e} = (2.40 \times 10^{-8})(T_t^{3/2}) \quad \text{g/cm}^3 \quad (1.3.12)$$

For a metal at 100 K,  $\rho_t/m_e = 6 \times 10^{-5} \text{ gm/cm}^3$ , which implies that the electrons in such a conductor follow the degenerate equation of state and that virtually all the cells in phase space are full. This accounts for the high conductivity of metals, since the saturation of phase space cells implies that free electrons cannot scatter off the

other particles in the metal (for in doing so they would have to move to a new cell in phase space, which is more than likely filled). Thus, they travel relatively unhindered through the conductor. In general, a totally degenerate gas proves to be an excellent conductor.

For temperatures on the order of  $10^7$  K which prevail in the center of the sun, the transition densities occur at about  $8 \times 10^2$  g/cm<sup>3</sup> which is significantly higher than we find in the sun. Thus, we may be assured that the ideal-gas law will be appropriate throughout the interior of the sun and most stars. However, white dwarfs do exceed the transition density for the temperatures we may expect in these stars. Therefore, we can expect a transition from the ideal-gas law which will prevail in the surface regions to total degeneracy in the interior. In this transition region the equation of state becomes more complex. A complete discussion of this region can be found in Cox and Giuli<sup>6</sup> and Chandrasekhar<sup>7</sup>. The basic philosophy is to write the equation of state in parametric form in terms of a degeneracy parameter  $y$ , where the equation of state becomes the ideal-gas law when  $y \ll 0$  and the equation of state approaches the totally degenerate equation of state if  $y \gg 0$ . This parametric form can be written as

$$\begin{aligned} \frac{\rho}{\mu_e} &= \left(\frac{4\pi}{h^3}\right) (2mkT)^{3/2} m_h F_{1/2}(\psi) \\ P_e &= \left(\frac{8\pi}{3h^3}\right) (2mkT)^{3/2} k T F_{3/2}(\psi) \\ F_n(\psi) &= \int_0^\infty \frac{x^n dx}{e^{x-\psi} + 1} \end{aligned} \tag{1.3.13}$$

In the transition zone between nonrelativistic and relativistic degeneracy, S. Chandrasekhar<sup>7</sup> also gives a parametric equation of state in terms of the "maximum" momentum  $p_0$  of the Fermi sea:

$$\begin{aligned} \frac{\rho}{\mu_e} &= \left(\frac{8\pi}{3h^3}\right) (m_e c)^3 m_h x^3 \\ P_e &= \frac{\pi}{3h^3} m_e^4 c^5 f(x) \\ f(x) &= x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x \end{aligned} \tag{1.3.14}$$

As  $x$  approaches zero, the nonrelativistic equation of state is obtained whereas as  $x$  approaches infinity, the fully relativistic limit is obtained. In the rare case where the gas occupies both transition regions at the same time, the equation of state becomes quite complicated. Refer to Cox and Giuli for a detailed description of this situation<sup>8</sup>.

Before leaving this discussion of the equation of state and degenerate matter, we want to explore some consequences of the most notable aspect of the degenerate equation of state. Nowhere in either the nonrelativistic or the relativistic degenerate equation of state does the temperature appear. This complete lack of temperature dependence implies a unique relationship between the pressure and the density.

Hydrostatic equilibrium [equation (1.2.28)] implies a relation among the pressure, mass, and radius. Since the mass, density, and radius are related by definition, these three relationships should allow us to find a unique relation between the mass of the configuration and its radius. Although a detailed investigation of the relation requires the solution of a differential equation coupled with some extremely nonlinear equations, we can get a sense of the mass-radius relation by considering the form of the equations that constrain the solution.

For spherical stars, hydrostatic equilibrium as expressed by equation (1.2.28) implies that

$$\frac{dP}{dr} \sim \frac{M(r)\rho}{r^2} \sim \frac{M(M/R^3)}{R^2} \quad (1.3.15)$$

Since we can also expect the pressure gradient to be proportional to  $P/R$ , the internal pressure in a star should vary as

$$P \sim \frac{M^2}{R^4} \quad (1.3.16)$$

For a totally degenerate gas, the equation of state requires that

$$P \sim \rho^{5/3} \sim \frac{M^{5/3}}{R^5} \quad (1.3.17)$$

Thus, we eliminate the pressure from these two expressions to get

$$R \sim M^{-1/3} \quad \text{or} \quad M \sim R^{-3} \quad (1.3.18)$$

We arrive at a curious result: As the mass of the configuration increases, the radius decreases. This situation, then, must prevail for white dwarfs. The more massive the white dwarf, the smaller its radius. In a situation where mass is added to a white dwarf, thereby causing its radius to decrease, the internal pressure must increase, which leads to an increase in the Fermi momentum  $p_0$ . Sooner or later the equation of state must change over to the fully relativistic equation of state. Here

$$P \sim \rho^{4/3} \sim \frac{M^{4/3}}{R^4} \quad (1.3.19)$$

If we again eliminate the pressure by using equation (1.3.16), then the radius also disappears and

$$M \sim \text{constant} . \quad (1.3.20)$$

Thus, for a fully relativistic degenerate gas, there is a unique mass for which the configuration is stable. Should mass be added beyond this point, the star would be forced into a state of unrestrained gravitational collapse. Much later we shall see that a further change in the equation of state, which occurs when the density approaches that of nuclear matter, can halt the collapse, allowing the formation of a neutron star. But for "normal" matter a limit is set by quantum mechanics, and this prevents the formation of white dwarfs with masses greater than about  $1.4 M_{\odot}$ . This is the limit found by S. Chandrasekhar in the late 1930s and for which he received the Nobel Prize in 1983. The configuration described by the fully relativistically degenerate equation of state is a strange one indeed, and we shall explore it in some detail later. For now, let us turn to the most basic assumptions that must be made for the study of stellar structure and to what they imply about the nature of stars.

## Problems

1. Consider a standard deck of 52 playing cards dealt into four hands of 13 cards each. If a given suit distribution within a hand represents a macrostate while a specific set of cards within a suit represents a microstate, find
  - a the number of possible macrostates for each hand,
  - b the number of microstates allowed for each macrostate, and
  - c the most probable macrostate.
  
2. Consider a space with three cells of size  $2h^3$ , and nine particles. Find the total number of macrostates, the total number of microstates, and the most probable macrostate, assuming the particles are
  - a "Maxwellons",
  - b fermions, and
  - c bosons.
  
3. Given that
 
$$dN = B \left( \exp \frac{w}{kT} + \phi \right)^{-1} dp_x dp_y dp_z$$
 and that  $w = (\mathbf{p}_x^2 + \mathbf{p}_y^2 + \mathbf{p}_z^2)/m$ , find an expression for  $B$  in terms of  $N$  for the cases where  $\phi = 0, \pm 1$ .
  
4. Derive the equation of state for a Fermi gas from first principles.
  
5. Given that  $f(x)$  is an analytic function in the interval  $0 \leq x \leq 4$ , show that  $f(x)$  can be represented in terms of the moments of the function  $M_i[f(x)]$ , where

### 1 Stellar Interiors

$$M_i[f(x)] \equiv \int_0^\infty x^i f(x) dx$$

6. If the pressure tensor  $\mathbf{P}$  has the form specified by equation (1.2.25), show that it can be rewritten as it appears in equation (1.2.24) (i.e., as the tensor operated on by the divergence operator in the third term on the left-hand side).
7. Show that the Virial theorem holds in the form given by equation (1.2.35) even if the forces of interaction include velocity dependent terms (i.e., such as Lorentz forces or viscous drag forces).
8. Show that the second velocity moment of the Boltzmann transport equation leads to an equation describing the conservation of energy.

### References and Supplemental Reading

1. Aller, L.H., *The Atmospheres of the Sun and Stars*, 2d ed., Ronald Press, New York, 1963, pp. 104, 108.
2. Ogorodnikov, K.F., *Dynamics of Stellar Systems*, Trans: J.B.Sykes, Ed. A.Beer, Macmillan, New York, 1965, p. 143.
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4. Landau, L.D., and Lifshitz, E.M., *Mechanics*, Trans: J.B.Sykes and J.S.Bell, Addison-Wesley, Reading, Mass., 1960, p. 147.
5. Collins, G.W., II, *The Virial Theorem in Stellar Astrophysics*, Pachart Publishing House, Tucson Ariz., 1978, p. 14.
6. Cox, J.P., and Giuli, R.T., *Principles of Stellar Structure*, Gordon and Breach, New York, 1968, p. 781.
7. Chandrasekhar, S., *An Introduction to Stellar Structure*, Dover, New York, 1957, p. 401.
8. \_\_\_\_\_, *ibid.*, p. 360.
9. Cox, J.P., and Giuli, R.T., *Principles of Stellar Structure* Gordon and Breach, New York, 1968, p. 812.

*1. Introduction and Fundamental Principles*

The reader may wish to consult a number of supplemental references to further understand the material in this chapter. Since the approach is basically that of statistical mechanics, any good book on that subject should enhance the readers understanding. Some examples are:

Reif, F., *Statistical Physics*, McGraw-Hill, New York, 1967.

Akhiezer, A.I., and Peletminskii, S.V.: *Methods of Statistical Physics*, Pergamon, New York, 1981.

Anderws, F.C., *Equilibrium Statistical Mechanics*, Wiley, New York, 1975.

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