2

Basic Assumptions, Theorems and Polytropes

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2.1 Basic Assumptions

Any rational structure must have a beginning, a set of axioms, upon which to build. In addition to the known laws of physics, we shall have to assume a few things about stars to describe them. It is worth keeping these assumptions in mind for the day you encounter a situation in which the basic axioms of stellar structure no longer hold. We have already alluded to the fact that a self-gravitating plasma will assume a spherical shape. This fact can be rigorously demonstrated from the nature of an attractive central force so it does not fall under the category of an axiom.

However, it is a result which we shall use throughout most of this book. A less obvious axiom, but one which is essential for the construction of the stellar interior, is that the density is a monotonically decreasing function of the radius. Mathematically, this can be expressed as

$$\rho(r) \le \langle\rho\rangle(r) \quad \text{for } r > 0 \ , \tag{2.1.1}$$

where

$$\langle\rho\rangle(r) \ / \ M(r)/[4\pi r^3/3] \ , \tag{2.1.2}$$

and M(r) is the mass interior to a sphere of radius r and is just $\int 4\,\pi\, r^2\, \rho\, dr$. In addition, we assume as a working hypothesis that the appropriate equation of state is the ideal-gas law. Although this is expressed here as an assumption, we shall shortly see that it is possible to estimate the conditions which exist inside a star and that they are fully compatible with the assumption.

It is a fairly simple matter to see that the free-fall time for a particle on the surface of the sun is about 20 min. This is roughly equivalent to the dynamical time scale which is the time scale on which the sun will respond to departures from hydrostatic equilibrium. Most stars have dynamical time scales ranging from fractions of a second to several months, but in all cases this time is a small fraction of the typical evolutionary time scale. Thus, the assumption of hydrostatic equilibrium is an excellent one for virtually all aspects of stellar structure. In Chapter 1 we developed an expression for hydrostatic equilibrium [(equation (1.2.28)], where the pressure gradient is proportional to the potential gradient and the local constant of proportionality is the density. For spherical stars, we may take advantage of the simple form of the gradient operator and the source equation for the gravitational potential to obtain a single expression relating the pressure gradient to M(r) and ρ.

The source equation for the gravitational potential field is also known as *Poisson's equation* and in general it is

$$\nabla^2\Omega = 4\pi G\rho \ , \tag{2.1.3}$$

which in spherical coordinates becomes

$$\frac{d}{dr}\left(r^2\,\frac{d\Omega}{dr}\right) = 4\pi G\rho r^2 \tag{2.1.4}$$

Integrating this over r, we get

$$\frac{d\Omega}{dr} = \frac{G}{r^2}\int_0^r 4\pi r^2 \rho\, dr = \frac{GM(r)}{r^2} \tag{2.1.5}$$

Replacing the potential gradient from equation (1.2.28), we have

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \tag{2.1.6}$$

This is the equation of hydrostatic equilibrium for spherical stars. Because of its generality and the fact that virtually no assumptions are required to obtain it, we can use its integral to place fairly narrow limits on the conditions that must prevail inside a star.

In equation (2.1.2), we introduced a new variable M(r). Note that its invocation is equivalent to invoking a conservation law. The conservation of mass basically requires that the total mass interior to r be accounted for by summing over the density interior to r. Thus,

$$M(r) = \int_0^r 4\pi r^2 \rho \, dr \tag{2.1.7}$$

or its differential form

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho \tag{2.1.8}$$

2.2 Integral Theorems from Hydrostatic Equilibrium

a Limits on State Variables

Following Chandrasekhar, [1] we wish to define a quantity $I_{\sigma,\nu}(r)$ which is effectively the σth moment of the mass distribution further weighted by $r^{-\nu}$. Specifically

$$I_{\sigma,\nu}(r) \equiv \frac{G}{4\pi} \int_0^r \frac{[M(r)]^\sigma}{r^\nu} \, dM(r) \tag{2.2.1}$$

There are quite a variety of physical quantities which can be related to $I_{\sigma,\nu}$. For example,

$$4\pi I_{11} = G \int_0^r M(r)\rho 4\pi r^2 \frac{dr}{r} \tag{2.2.2}$$

is just the absolute value of the total gravitational energy of the star.

We can use this integral quantity to place limits on physical quantities of interest if we replace ρ by <ρ> as defined by equation (2.1.2). Since

$$r^{\nu} = \left[\frac{M(r)}{\frac{4}{3}\pi\langle\rho\rangle(r)}\right]^{\nu/3} \tag{2.2.3}$$

we may rewrite $I_{\sigma,\nu}$ as

$$I_{\sigma,\nu}(r) = \frac{G}{4\pi}\int_0^r [M(r)]^{\sigma-\nu/3}\left(\frac{4\pi}{3}\right)^{\nu/3}\langle\rho\rangle(r)^{\nu/3}\,dM(r) \tag{2.2.4}$$

Now since our assumption of the monotonicity of ρ requires $\rho_c \geq \langle\rho\rangle(r) \geq \rho(r)$, we can obtain an inequality to set limits on $I_{\sigma,\nu}$. Namely,

$$\frac{G}{4\pi}\left[\frac{4\pi}{3}\right]^{\nu/3}\rho_c^{\nu/3}\frac{M^{\sigma+1-\nu/3}(r)}{(\sigma+1-\nu/3)} \geq I_{\sigma,\nu}(r) \geq \frac{G}{4\pi}\left[\frac{4\pi}{3}\right]^{\nu/3}<\rho>^{\nu/3}(r)\frac{M^{\sigma+1-\nu/3}(r)}{(\sigma+1-\nu/3)} \tag{2.2.5}$$

Now let us relate <***P***>, <***T***>, and <***g***> to $I_{\sigma,\nu}$, where these quantities are defined as

$$\langle P\rangle \equiv \int_0^M P(r)\frac{dM(r)}{M}$$
$$\langle T\rangle \equiv \int_0^M T(r)\frac{dM(r)}{M}$$
$$\langle g\rangle \equiv \int_0^M g(r)\frac{dM(r)}{M} \tag{2.2.6}$$

Making use of the result that the surface pressure and temperature are effectively zero compared to their internal values, we can eliminate the temperature by using the ideal-gas law, integrate the first two members of equations (2.2.6) by parts and eliminate the pressure gradient by utilizing hydrostatic equilibrium. We obtain

$$\langle P\rangle = \frac{I_{2,4}(R)}{M}$$
$$\langle T\rangle = \frac{4\pi\mu m_h}{3k}\frac{I_{1,1}(R)}{M}$$
$$\langle g\rangle = \frac{4\pi I_{1,2}(R)}{M} \tag{2.2.7}$$

The last of these expressions comes immediately from the definition of ***g***. Applying the inequality [(equation (2.2.5)], we can immediately obtain lower limits for these quantities of

$$\begin{aligned}\langle P\rangle &\geq \frac{3GM^2}{20\pi R^4} = 5.4\times 10^8\left(\frac{M}{M_\odot}\right)^2\left(\frac{R_\odot}{R}\right)^4 \quad \text{atm}\\ \langle T\rangle &\geq \frac{G\mu m_h M}{5kR} = 4.61\times 10^6\,\mu\left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right) \quad \text{K}\\ \langle g\rangle &\geq \frac{3GM}{4R^2} = 2.05\times 10^4\left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right)^2 \quad \text{cm/s}^2\end{aligned} \tag{2.2.8}$$

Since these theorems apply for any gas sphere in hydrostatic equilibrium where the ideal-gas law applies, we can use them for establishing the range of values to be expected in stars in general. In addition, it is possible to use the other half of the inequality to place upper limits on the values of these quantities at the center of the star.

b β^* Theorem and Effects of Radiation Pressure

We have consistently neglected radiation pressure throughout this discussion and a skeptic could validly claim that this affects the results concerning the temperature limits. However, there is an additional theorem, also due to Chandrasekhar[1] (p.73), which places limits on the effects of radiation pressure. This theorem is generally known as the β^* theorem. Let us define β as the ratio of the gas pressure to the total pressure which includes the radiation pressure. The radiation pressure for a photon gas in equilibrium is just $P_r = aT^4/3$. Combining these definitions with the ideal-gas law, we can write

$$\begin{aligned}P_g &= \beta P_T = \left[\frac{3}{a}\left(\frac{k}{\mu m_h}\right)^4\frac{1-\beta}{\beta}\right]^{1/3}\rho^{4/3}\\ P_r &= (1-\beta)P_T = \frac{aT^4}{3}\\ T &= \left[\frac{3k\,(1-\beta)}{\mu a m_h \beta}\right]^{1/3}\rho^{1/3}\\ P_c &= \frac{P_{g,c}}{\beta_c} = \frac{1}{\beta_c}\left[\frac{3}{a}\left(\frac{k}{\mu m_h}\right)^4\frac{1-\beta_c}{\beta_c}\right]^{1/3}\rho_c^{4/3}\end{aligned} \tag{2.2.9}$$

Using the integral theorems to place an upper limit on the central pressure, we get

$$P_c \leq \frac{1}{2}G\left(\frac{4\pi}{3}\right)^{1/3}\rho_c^{4/3}M^{2/3} \tag{2.2.10}$$

Equation (2.2.10), when combined with the last of equations (2.2.9) and solved for *M*, yields

$$M \geq \left(\frac{6}{\pi}\right)^{1/2}\left[\frac{1-\beta_c}{\beta_c^4}\left(\frac{k}{\mu m_h}\right)^4\frac{3}{a}\right]^{1/2} G^{-3/2} \tag{2.2.11}$$

Now we define β^* to be the value of β which makes Equation (2.2.11) an equality, and then we obtain the standard result that

$$\frac{1-\beta^*}{(\beta^*)^4} \geq \frac{1-\beta_c}{\beta_c^4} \tag{2.2.12}$$

Since $(1-\beta)/\beta^4$ is a monotone increasing function of $(1-\beta)$,

$$1-\beta^* \geq 1-\beta_c = \frac{P_{r,c}}{P_T} \tag{2.2.13}$$

Equation (2.2.11) can be solved directly for M in terms of β^* and thus it places limits on the ratio of radiation pressure to total pressure for stars of a given mass. Chandrasekhar[1] (p.75) provides the brief table of values shown in Table 2.1.

Table 2.1 Stellar Mass vs. $1 - \beta^*$

$1-\beta^*$	$(M/M_\odot)\mu^2$
0.01	0.56
0.02	0.81
0.10	2.14
0.20	3.83
0.30	6.12
0.40	9.62
0.50	15.50
0.60	26.52
0.70	50.93
0.80	122.50
0.90	520.7
1.00	∞

As we shall see later, m is typically of the order of unity (for example μ is ½ for pure hydrogen and 2 for pure iron). It is clear from Table 2.1, that by the time that radiation pressure accounts for half of the total pressure, we are dealing with a very massive star indeed. However, it is equally clear that the effects of radiation pressure must be included, and they can be expected to have a significant effect on the structure of massive stars.

2.3 Homology Transformations

The term *homology* has a wide usage, but in general it means "proportional to" and is denoted by the symbol ~ **.** One set is said to be *homologous* to another if the two can be put into a one-to-one correspondence. If every element of one set, say z_i, can be identified with every element of another set, say z'_i, then $z_i \sim z'_i$ and the two sets are homologous. Thus a homology transformation is a mapping which relates the elements of one set to those of another. In astronomy, the term *homology* has been used almost exclusively to relate one stellar structure to another in a special way.

One can characterize the structure of a star by means of the five variables P(r), T(r), *M*(r), μ(r), and ρ(r) which are all dependent on the position coordinate r. In our development so far, we have produced three constraints on these variables, the ideal-gas law, hydrostatic equilibrium, and the definition of *M*(r). Thus specifying the transformation of any two of the five dependent variables and of the independent variable r specifies the remaining three. If the transformations can be written as simple proportionalities, then the two stars are said to be homologous to each other. For example, if

$$r' = C_1 r \qquad \rho'(r') = C_2\rho(r) \qquad \xi'(r') = C_3\xi(r) \tag{2.3.1}$$

then

$$\zeta'(r') = C_4\zeta(r) \qquad \eta'(r') = C_5\eta(r) \qquad \chi'(r') = C_6\chi(r) \tag{2.3.2}$$

where ξ, ζ, η, and χ stand for any of the remaining structure variables. However, because of the constraints mentioned above, C_4, C_5, and C_6 are not linearly independent but are specified in terms of the remaining C's. Consider the definition of *M*(r) and a homology transformation from r → r'. Then

$$\frac{M'(r')}{M(r)} = \frac{\int_0^{r'} 4\pi(x')^2\rho'(x')\,dx'}{\int_0^{r} 4\pi x^2\rho(x)\,dx} = C_2C_1^3 \tag{2.3.3}$$

In a similar manner, we can employ the equation of hydrostatic equilibrium to find the homology transformation for the pressure P, since

$$\frac{P'(r')}{P(r)} = \frac{\int_0^{r'} [GM'(x')\rho'(x')/x'^2]\,dx'}{\int_0^{r} [GM(x)\rho(x)/x^2]\,dx} = C_2^2C_1^2 \tag{2.3.4}$$

If we take μ to be the chemical composition m, then the remaining structure variable is the temperature whose homology transformation is specified by the ideal-gas law as

$$\frac{P'(r')}{P(r)} = \frac{\rho'(r')kT'(r')/\mu'(r')}{\rho(r)kT(r)/\mu(r)} = C_2^2 C_1^2 \tag{2.3.5}$$

so that

$$\frac{T'(r')}{T(r)} = C_3 C_2 C_1^2 \tag{2.3.6}$$

Should we take ξ to be T, then the homology transform for μ is specified and is

$$\frac{\mu'(r')}{\mu(r)} = \frac{C_3}{C_1^2 C_2} \tag{2.3.7}$$

We can use the constraints specified by equations (2.3.3), (2.3.4), and (2.3.6) and the initial homology relations [equation (2.3.1)] to find how the structure variables transform in terms of observables such as the total mass M and radius R. Thus,

$$\begin{aligned}
\frac{\rho'(r')}{\rho(r)} &= \frac{M'}{M}\left(\frac{R}{R'}\right)^3 \\
\frac{P'(r')}{P(r)} &= \left(\frac{M'}{M}\right)^2\left(\frac{R}{R'}\right)^4 \\
\frac{T'(r')}{T(r)} &= \frac{\mu' M' R}{\mu M R'}
\end{aligned} \tag{2.3.8}$$

Since homology transformations essentially represent a linear scaling from one structure to another, it is not surprising that the dependence on mass and radius is the same as implied by the integral theorems [equations (2.2.8)].

The primary utility of homology transformations is that they provide a "feel" for how the physical structure variables change given a simple change in the defining parameters of the star, "all other things being equal." An intuitive feel for the behavior of the state variables P, T, and p which result from the scaling of the mass and radius is essential if one is to understand stellar evolution. Consider the homologous contraction of a homogeneous uniform density mass configuration. Here the total mass and composition remain constant, and we obtain a very specific homology transformation

$$\frac{\rho}{\rho_0}=\left(\frac{R_0}{R}\right)^3 \qquad \frac{P}{P_0}=\left(\frac{R_0}{R}\right)^4 \qquad \frac{T}{T_0}=\frac{R_0}{R} \tag{2.3.9}$$

which is known as *Lane's Law*[1] (p.47) and has been thought to play a role in star formation. In addition, certain phases of stellar collapse have been shown to behave homologously. In these instances, the behavior of the state variables is predictable by simple homology transformations in spite of the complicated detailed physics surrounding these events.

2.4 Polytropes

We have progressed about as far as we can in setting conditions for stellar structure with the assumptions that we made. It is now necessary to add a constraint on the structure. Physically, the logical arenas to search for such constraints are energy production and energy flow, and we shall do so in later chapters. However, before we enter those somewhat complicated domains, consider the impact of a somewhat ad hoc relationship between the pressure and the density. This relationship has its origins in thermodynamics and results from the notion of polytropic change. This gives rise to the polytropic equation of state

$$P(r) = \mathbf{K}\rho(r)^{(n+1)/n} \tag{2.4.1}$$

where n is called the polytropic index. Clearly, an equation of state of this form, when coupled with the equation of hydrostatic equilibrium, will provide a single relation for the run of pressure or density with position. The solution of this equation basically solves the fundamental problem of stellar structure insofar as the equation of state correctly represents the behavior of the stellar gas. Such solutions are called polytropes of a particular index n.

Many astrophysicists feel that the study of polytropes is of historical interest only. While it is true that the study of polytropes did develop early in the history of stellar structure, this is so because polytropes provide significant insight into the structure and evolution of stars. The motivation comes from the observation that ideal gases behave in a certain way when they change in an adiabatic manner. It is a generalization of this behavior which is characterized by the polytropic equation of state. Later we shall see that when convection is established in the interior of a star, it is so efficient that the resultant temperature gradient is that of an adiabatic gas responding to hydrostatic equilibrium. Such a configuration is a polytrope. We have already seen that the degenerate equations of state have the same form as the polytropic equation of state, and so we might properly expect that degenerate configurations will be well represented by polytropes. In addition, we shall find that in massive stars where the pressure is dominated by the pressure of radiation, the equation of state is essentially that of a photon gas in statistical equilibrium and that equation of state is also polytropic. The simple nature of polytropic structure and its correspondence to many physical stars provides a basis for incorporating additional effects (such as rotation) in a semi-analytical manner and thereby offers insight into the nature of the effects in real stars. Thus, for providing insight into the structure and behavior of real stars, an understanding of polytropes is essential. However, even beyond the domain of stellar astrophysics, polytropes find many applications. Certain problems in stellar dynamics and galactic structure can be described by polytropes, and the polytropic equation of state has even been used to represent the density distribution of dark matter surrounding galaxies. But with the applications to stars in mind, let us consider the motivation for the polytropic equation of state.

a Polytropic Change and the Lane-Emden Equation

From basic thermodynamics we learn that the infinitesimal change in the heat of a gas $đQ$ can be related to the change in the internal energy dU and the work done on the gas so that

$$đQ = dU + PdV = \frac{\partial U}{\partial T}dT + PdV \tag{2.4.2}$$

The strange-looking derivative $đ$ is known as a *Pfaffian derivative*, and its most prominent property is that it is not an exact differential. A complete discussion of the mathematical properties is given by Chandrasekhar[1] (p.17). The ideal-gas law can be stated in its earliest form as PV = RT, which leads to

$$PdV+VdP = RdT \tag{2.4.3}$$

where R is the gas constant and V is the specific volume (i.e., the volume per unit mass). Now let us define the specific heat at constant a C_α as

$$\left.\left[\frac{đQ}{dT}\right]\right|_{\alpha = \mathrm{const}} \equiv C_\alpha \tag{2.4.4}$$

Here, the differentiation is done in such a way that α remains constant. Thus $(dU/dT)|_V$ is the specific heat, C_V, at constant volume. Using equation (2.4.3) to eliminate PdV in equation (2.4.2) we get

$$C_P = C_V + R \tag{2.4.5}$$

where C_P is the specific heat at constant pressure.

With this notion that $(đQ/dT)|_\alpha$ is the specific heat at constant α, we make the generalized definition of polytropic change to be

$$\frac{dQ}{dT} = C \tag{2.4.6}$$

where C is some constant. Using equations (2.4.2), and (2.4.3) and the definition of C we can write

$$T^{C_V - C}V^{C_P - C_V} = \text{const} \tag{2.4.7}$$

Now for an ordinary gas it is common to define the ratio of specific heats (C_P/C_V) as γ. In that same spirit, we can define a polytropic gamma as

$$\gamma' = \frac{C_P - C}{C_V - C} \tag{2.4.8}$$

By use of the ideal-gas law, we can write

$$P = \text{const} \times V^{-\gamma'} = (\text{const})\left(\frac{k}{\mu m_h}\right)\rho^{\gamma'} = K\rho^{(n+1)/n} \tag{2.4.9}$$

Thus, we can relate the specific heat C associated with polytropic change to the polytropic index *n* to be found in the polytropic equation of state [equation (2.4.1)]. So

$$n = 1/(\gamma'-1) \tag{2.4.10}$$

If C = 0, then the general relation describes where the change in the internal energy is equal to the work done *on* the gas [see equation (2.4.2)], which means the gas behaves adiabatically. If C = 4, then the gas is isothermal.

The polytropic equation of state provides us with a highly specific relationship between P and ρ. However, hydrostatic equilibrium also provides us with a specific relationship between P and ρ, and we may use the two to eliminate the pressure P, thereby obtaining an equation in ρ alone which describes the run of density throughout the configuration. Differentiating equation (2.1.6) with respect to r and eliminating P by means of the polytropic equation of state, we get

$$\frac{d}{dr}\left[\frac{Kr^2(n+1)}{n\rho^{(n-1)/n}}\frac{d\rho}{dr}\right] = -4\pi r^2 G\rho \tag{2.4.11}$$

This nonlinear second-order differential equation for the density distribution is subject to the boundary conditions $\rho(0) = \rho_c$ and $\rho(R) = 0$. Or to put it another way, the radius of the configuration is defined to be that value of r for which $\rho = 0$. The only free parameters in the equation are the polytropic index (*n*) and the parameter K and any solution to such an equation is called a *polytrope*. The parameter K is related to the total mass of the configuration. In addition, the equation is generally known as the *Lane-Emden Equation*. However, in this case, we have written it in physical variables. During the nineteenth and early twentieth century, a considerable effort was expended in the solution of this equation for various values of the polytropic index (*n*). If one is going to investigate the general solution-set of any equation, it is usually a good idea to express the equation in a dimensionless form. This can be done to equation (2.4.11) by transformation to the so-called Emden variables given by

$$\rho = \lambda\theta^n \qquad r = \alpha\xi \tag{2.4.12}$$

where,

$$\alpha = \left[\frac{(n+1)K\lambda^{1/n-1}}{4\pi G}\right]^{1/2} \tag{2.4.13}$$

Here λ is just a scaling parameter useful for keeping track of the units of ρ and plays no role in the resulting equation. It is clear that ξ is just a scaled, dimensionless radius while θ's meaning is rather more obscure. While θ is dimensionless by virtue of using λ to absorb the units of ρ, it does vary as $\rho^{(1/n)}$ and is the normalized ratio of P/ρ. If we make the substitutions indicated by equation (2.4.12) we obtain the more familiar form of the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{2.4.14}$$

By picking **K** and *n* we can transform any solutions of eq (2.4.14) and obtain the solution for the polytrope of a given mass M and index *n* in terms of the run of physical density with position. The non-linear nature of the transformation has had the advantage that the boundary conditions of the physical equation can easily be written as initial conditions at ξ = 0. The utility of λ now becomes clear as we can scale $\theta(0)$ to be 1 so that

$$\theta(0) = 1 \qquad \lambda = \rho_c \qquad \left.\frac{d\theta}{d\xi}\right|_{\xi=0} = 0 \tag{2.4.15}$$

The last initial condition comes from hydrostatic equilibrium. As r → 0, M(r) → 0 as r^3 and $\rho \to \rho_c$. Thus it is clear from equation (2.1.6) that dP/dr → 0 as well. This implies that $d\theta/d\xi \to 0$ as $\xi \to 0$.

In principle, we are now prepared to solve the Lane-Emden Equation for any polytropic index *n*. Unfortunately, only three analytic solutions exist, and they are for

n = 0, 1, and 5. None of these correspond to particularly interesting physical situations, but in the hopes of learning something about the general behavior of polytropic solutions we give them:

$$\begin{aligned} \theta_0(\xi) &= 1 - \frac{\xi^2}{6} && n = 0 \\ \theta_1(\xi) &= \frac{\sin \xi}{\xi} && n = 1 \\ \theta_5(\xi) &= \left(1 + \frac{\xi^2}{3}\right)^{-1/2} && n = 5 \end{aligned} \tag{2.4.16}$$

For n = 0, we see that the solution is monotonically decreasing toward the surface which is physically reasonable. This is also true for n = 1, and n = 5 although the rate of decline is slower. Indeed, the n = 5 case only, θ asymptotically approaches zero from arbitrarily large ξ. If we denote the value of ξ for which θ goes to zero as ξ_1, then

$$\xi_1 = \begin{cases} \sqrt{6} & n = 0 \\ \pi & n = 1 \\ \infty & n = 5 \end{cases} \tag{2.4.17}$$

The value of r which corresponds to ξ_1 is clearly the radius R of the configuration. For other values of the polytropic index n it is possible to develop a series solution which is useful for starting many numerical methods for the solution. The first few terms in the solution are

$$\theta_n = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \left(\frac{8n^2 - 5n}{15{,}120}\right)\xi^6 + \cdots + \tag{2.4.18}$$

b Mass-Radius Relationship for Polytropes

For these solutions to be of any use to us, we must be able to relate them to a configuration having a specific mass and radius. We have already indicated how the radius is related to ξ_1 and α, which really means that the mass is related to n and **K**. Now let us turn to the relationship between the mass of the configuration and the parameters of the polytrope. By using the definition of M(r) and the Lane-Emden equation, (2.4.14), to eliminate θ_n, we can write

$$M(\xi_1) = \int_0^{R/\alpha} 4\pi r^2 \rho \, dr = 4\pi\alpha^3\lambda \int_0^{\xi_1} \xi^2\theta^n \, d\xi = -4\pi\alpha^3\rho_c\left(\xi_1^2 \frac{d\theta}{d\xi}\bigg|_{\xi_1}\right) \tag{2.4.19}$$

for the total mass. Using R = $\alpha\xi_1$ to eliminate α for the expression for M, we can obtain a mass-radius relation for any polytrope.

$$GM^{(n-1)/n} R^{(3-n)/n} = -K(n+1)\left[(4\pi)^{-1/n}\right]\left[\xi_1^{(n+1)/(n-1)}\left(\frac{d\theta}{d\xi}\right)\bigg|_{\xi_1}\right]^{\frac{(n-1)}{n}} \qquad (2.4.20)$$

For a given configuration, equation (2.4.20) can be used to determine **K** since everything else on the right-hand side depends on only the polytropic index *n*. Thus, for a collection of polytropic model stars we can write the mass-radius relation as

$$M^{(n-1)/n} R^{(3-n)/n} = (\mathrm{const})(n) \ . \qquad (2.4.21)$$

c Homology Invariants

We can apply what we have learned about homology transformations to polytropes. In general, if $\theta_n(\xi)$ is a solution of the Lane-Emden equation, then $A^{2/(n-1)}\theta_n(A\xi)$ is also a solution (for a proof see Chandrasekhar[6]). Here A is an arbitrary constant, so $A\xi$ is clearly a homology transformation of ξ. This produces an entire family of solutions to the Lane-Emden equation, and it would be useful if we could obtain a set of solutions which contained all the homology solutions. To do this, we must find a set of variables which are invariant to homology transformations. Chandrasekhar[1] (p.105) suggests the following variables

$$u \equiv \frac{d\ln[M(r)]}{d\ln r} = \frac{3\rho(r)}{<\rho(r)>} = \frac{-\xi\theta^n}{d\theta/d\xi}$$

$$(2.4.22)$$

$$(n+1)v \equiv -\frac{d\ln[P(r)]}{d\ln r} = +\frac{3}{2}\frac{[GM(r)/r]}{[(3/2)kT/\mu m_h]} = -(n+1)\xi\theta^{-1}\frac{d\theta}{d\xi}$$

as representing a suitable set of variables which are invariant to homology transformations. The physical interpretation of u is that it is 3 times the ratio of the local density to the local mean density, while (*n*+1)v is simply 1.5 times the ratio of the local gravitational energy to the local internal energy. In general, these quantities will remain invariant to any change in the structure which can be described by a homology transformation. We can use these variables to rewrite the Lane Emden equation so as to obtain all solutions which are homologous to each other.

$$\frac{u}{v}\frac{dv}{du} = -\frac{u+v-1}{u+nv-3} \tag{2.4.23}$$

Not all solutions to this equation are physically reasonable. For instance, at $\xi = 0$ we must require that $\theta(\xi)$ remain finite. One can show by substituting into hydrostatic equilibrium as expressed in Emden variables, that $d\theta/d\xi = 0$ at $\xi = 0$. This requires that at the center of the polytrope the values [u=3, v=0] set the initial conditions for the unique solution meeting the minimal requirements for being a physical solution. These solutions are known as the *E-solutions* and we have already given a series expansion for the θ_E solution in equation (2.4.18). By substituting this series into the equations for u and v, and expanding by the binomial theorem we obtain the following series solutions for u and v:

$$\begin{aligned} v &= \frac{\xi^2}{3}\left(1 - \frac{3n-5}{30}\xi^2 + \frac{12n^2-39n+35}{1260}\xi^4 + \cdots +\right) \\ u &= 3 - \frac{n\xi^2}{5} + \frac{(19n^2-25n)\xi^4}{1050} \\ &\quad - \frac{(472n^3-1275n^2+875n)\xi^6}{283{,}500} + \cdots + \end{aligned} \tag{2.4.24}$$

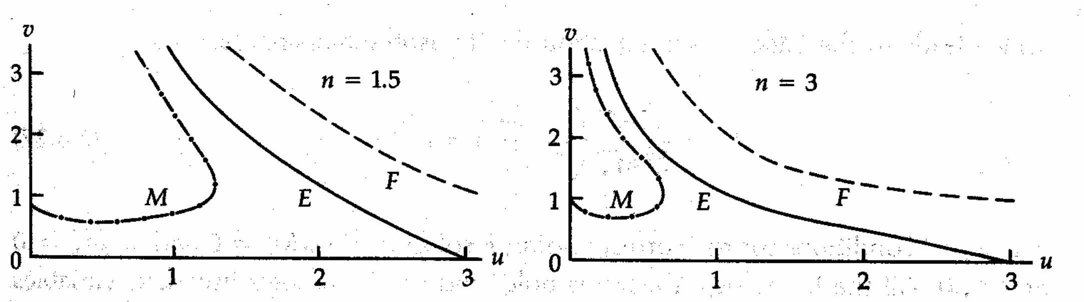

Figure 2.1 shows the solution for two common polytropes with physical interpretations. The solid lines represent the E-solutions which satisfy hydrostatic equilibrium at the origin. The dashed and dotted lines depict samples of the F- and M- solutions respectively. While these solutions do not satisfy the condition of hydrostatic equilibrium at the center of the polytrope, they may represent valid solutions for stars composed of multiple polytropes joined in the interior. The solution reaching the center must always be an E-solution. The polytrope with $n = 1.5$ represents the solution for a star in convective equilibrium, while the $n = 3$ polytrope solution is what is expected for a star dominated by radiation pressure.

As with the θ_E series, we may find the initial values for the numerical solution of the Lane-Emden equation and obtain the solution for a polytrope of any index which also satisfies hydrostatic equilibrium at its center. At the other end of the physical solution space, as $\xi \rightarrow \xi_1$, $\theta \rightarrow 0$, but the derivative of θ will remain finite. Thus as $u \rightarrow 0$, $v \rightarrow 4$. The part of solution space which will be of physical interest will then be limited to $u \geq 0$, $v \geq 0$. Figure 2.1 shows the solution set for two polytropic examples including the E-solutions.

d Isothermal Sphere

So far we have said nothing about what happens when the equation of state is essentially the ideal-gas law, but for various reasons the temperature remains constant throughout the configuration. Such situations can arise. For example, if the thermal conductivity is very high, the energy will be carried away rapidly from any point where an excess should develop. Such a configuration is known as an isothermal sphere, and it has a characteristic structure all its own. We already pointed out that an isothermal gas may be characterized by a polytropic C = 4. A brief perusal of equations (2.4.8) and (2.4.10) will show that this leads to a polytropic index of $n = 4$ and some problems with the Emden variables. Certainly the Lane-Emden equation in physical variables [equation (2.4.11)] is still valid since it involves only the hydrostatic equilibrium and the polytropic equation of state. However, we must investigate its value in the limit as $n \rightarrow 4$. Happily, the equation is well behaved in that limit, and we get

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\rho}{dr}\right) = -\frac{4\pi r^2 G\rho}{K} \tag{2.4.25}$$

However, some care must be exercised in transforming to the dimensionless Emden variables since the earlier transformation will no longer work. The traditional transformation is

$$r = \alpha\xi \qquad \rho = \lambda e^{-\psi} \qquad \alpha = \left(\frac{K}{4\pi G\lambda}\right)^{1/2} \tag{2.4.26}$$

which leads to the Lane-Emden equation for the isothermal sphere

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = e^{-\psi} \tag{2.4.27}$$

The initial conditions for the corresponding E solution are $\psi(0) = 0$ and $d\psi/d\xi q = 0$ at $\xi = 0$. All the homology theorems hold, and the homology invariant variables u and v have the same physical interpretation and initial values. In terms of these new Emden variables, they are

$$u = \frac{\xi e^{-\psi}}{d\psi/d\xi} \qquad v = \xi\frac{d\psi}{d\xi} \tag{2.4.28}$$

and the Lane-Emden equation in u and v is only slightly modified to account for the isothermal condition.

$$\frac{u}{v}\frac{dv}{du} = -\frac{u-1}{u+v-3} \tag{2.4.29}$$

The solution to this equation in the u-v plane is unique and is shown in Figure 2.2. In the vicinity of $\xi = 0$, ψ can be expressed as

$$\psi = \frac{\xi^2}{6} - \frac{\xi^4}{120} + \frac{\xi^6}{1890} - \frac{61\xi^8}{1{,}632{,}960} + \cdots + \tag{2.4.30}$$

which leads to the following expansions for the homology invariants u and v as given by equations (2.4.28).

$$u = 3 - \frac{\xi^2}{5} + \frac{19\xi^4}{1050} - \frac{118\xi^6}{70{,}875} + \cdots +$$
$$v = \left(\frac{\xi^2}{3}\right)\left(1 - \frac{\xi^2}{10} + \frac{\xi^4}{105} - \cdots +\right) \tag{2.4.31}$$

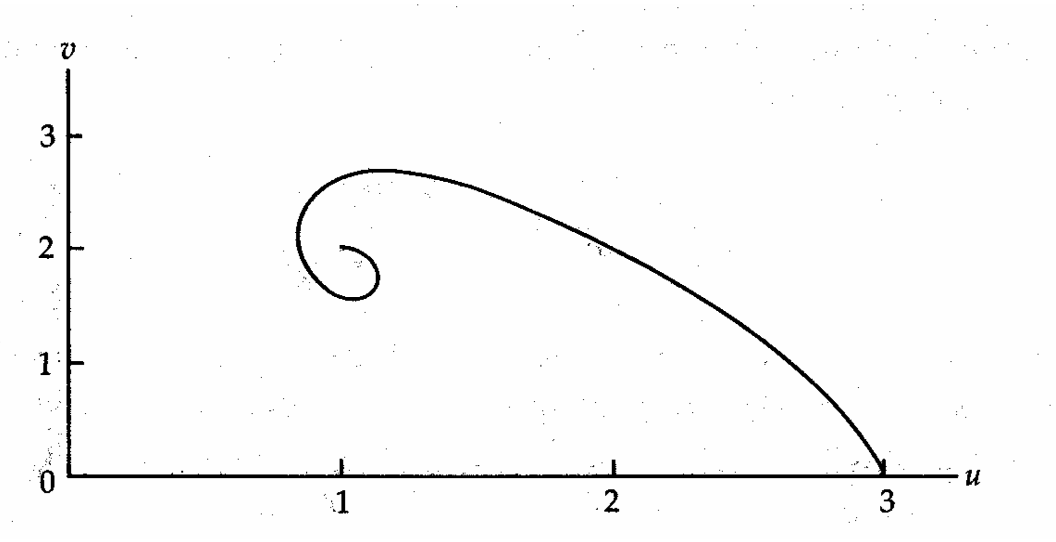

Figure 2.2 shows the solution for the isothermal sphere in the u-v plane. The solution is unique.

The physical importance of the isothermal sphere is widespread having applications from stellar cores to galaxy structure. So it is worthwhile to emphasize one curious aspect of the structure. Direct substitution of a density dependence with the form $\rho(r) \sim r^{-2}$ into equation (2.4.25) shows that such a density law will satisfy hydrostatic equilibrium at all points within an isothermal sphere. Thus, $\rho(r) \sim r^{-2}$ is often used to describe the radial density variation in spherically symmetric regions which are assumed to be isothermal.

e Fitting Polytropes Together

As we shall see later, many stars, including those on the main sequence, can be reasonably represented by a combination of polytropes where the local value of the polytropic index is chosen to reflect the physical constraints placed on the star by the mode of energy transport or possibly the equation of state. Thus, it is useful to understand what conditions must hold where the polytropes meet. Let us consider a simple star composed of a core and an envelope having different polytropic indices (see Figure 2.3).

Now let q be the fraction of the total mass in the core, n the polytropic index of the core, and m the total mass of the core. Physically, we must require that the pressure and density be continuous across the boundary. This implies that u and v are continuous across the boundary between the two polytropes. Since the initial conditions at the center of the core must be $u = 3$, $v = 0$, the core solution must be an E solution for the core index n. The envelope solution will not, in general, be an E solution; but as long as the central point ($u=3$, $v=0$) is not encountered, there is no violation of hydrostatic equilibrium by such a solution. Thus one can construct a reasonable model by proceeding outward along the core solution until the mass of the core is reached. This defines the fitting point in the u-v plane. One then searches the F or M solutions which meet the core solution at the fitting point, to ensure continuity of P and ρ across the boundary. There will be many solutions corresponding to different values of the polytropic index of the envelope. Picking one such solution, one continues with this solution until ξ_1 is reached, at which point $M(\xi_1)$ should equal m/q. If it does not, then there is no solution for that value of the polytropic index of the envelope and another solution at the fitting point should be chosen.

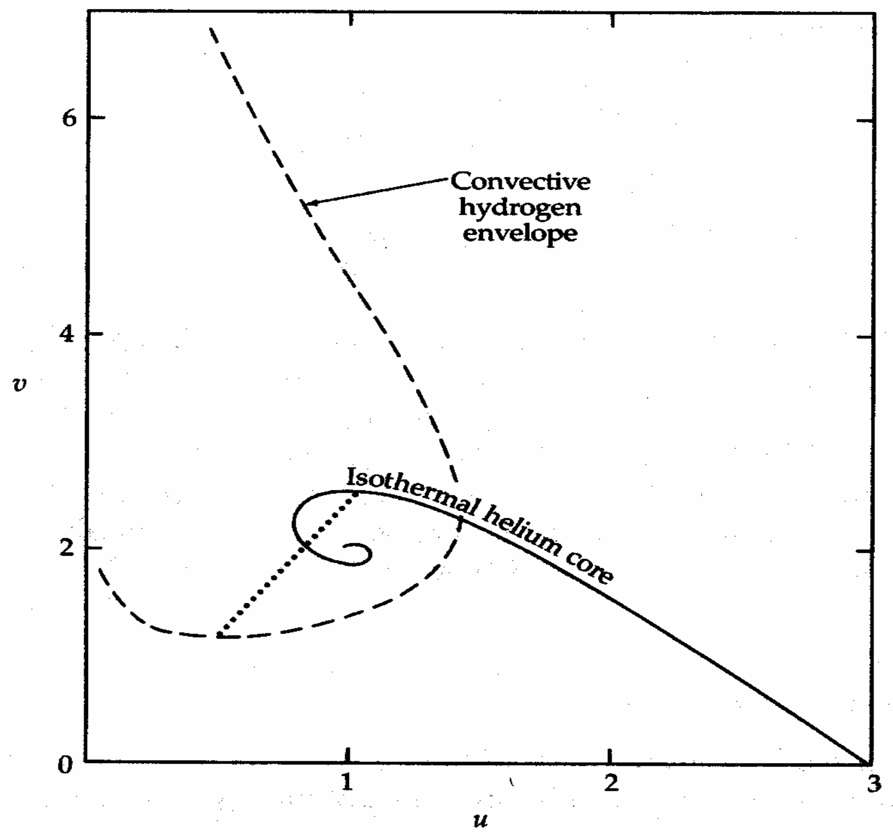

Figure 2.3 represents a model star composed of two polytropes. The outer convective hydrogen envelope can be represented by a polytrope of index $n = 1.5$, while the helium core is isothermal. The discontinuous change in u and v resulting from the change in chemical composition can be seen as a jump from the isothermal core solution toward the origin and the appropriate M-solution for the envelope. Such a model can be expected to qualitatively represent the evolved phase of a red giant.

The techniques of J.L. Lagrange known as *variation of parameters* can be utilized to convert an error on the mass at ξ_1, $dm(\xi_1)$, to a correction in the polytropic index δn_e of the envelope solution. Any solution which satisfies the continuity conditions and the constraints set by the core mass and mass fraction is unique. In addition, it is possible to allow for a discontinuity in the chemical composition at the boundary by permitting a discontinuity in the density such that momentum conservation is maintained across the boundary. That is, pressure equilibrium must be maintained across the boundary. From the ideal-gas law, the ratio of the density in the envelope to that of the core is

$$\frac{\rho_e}{\rho_c} = \frac{\mu_e}{\mu_c} \tag{2.4.32}$$

This is equivalent to specifying a jump in u and (n + 1)v by the ratio of the mean molecular weights of the core and envelope. Thus the fitting point, when it is reached, is displaced toward the origin in u and (n + 1)v by the ratio of the mean molecular weights of the envelope and core. This displaced point in the u-v plane is the new point from which the solution is to be continued (see Figure 2.3). The solution is then completed as in the previous instance.

By making use of polytropic solutions, it is possible to represent stars with convective cores and radiative envelopes with some accuracy and to get a rough idea of the run of pressure, density, and temperature throughout the star. Polytropes are useful in determining the effects of the buildup of chemical discontinuities as a result of nuclear burning. As mentioned earlier, very massive stars are radiation-dominated and are quite accurately represented by polytropes of index $n = 3$ ($\gamma' = 4/3$). Polytropes often can be used as an initial model which is then perturbed to approximate a given physical situation. For relatively little effort, polytropic models can provide substantial insight into the behavior of stars in response to various changes in physical conditions. We obtain this insight at a relatively low cost. To do significantly better, we must do much more. We will have to know, in some detail, how energy is transported throughout the star. But before we can do that, we must understand the detailed structure of the gas so that we can understand the properties which impede that flow of energy.

Problems

1. Use the integral theorems of Chandrasekhar to place limits on the central temperature of a star of given mass M.

2. Estimate the mass of a white dwarf at which the relativistic degenerate equation of state becomes essential for representing its structure.

3. Prove that all solutions to the Lane-Emden equation which remain finite at the origin ($\xi = 0$), must, of necessity, have

$$\left.\frac{d\theta}{d\xi}\right|_{\xi = 0} = 0$$

4. Show that the mass interior to ξ [that is, $M(\xi)$] in an isothermal sphere is given by

$$M(\xi) = 4\pi\alpha^3\lambda\xi^2 \frac{d\psi}{d\xi}$$

5. Find a series solution for the Lane-Emden Equation in the vicinity of $\xi = 0$, subject to the boundary condition that $(d\theta/d\xi)\big|_{\xi=0}$ be zero. This solution should have an accuracy of $O(\xi^{12})$.

6. Find a series solution for the isothermal sphere subject to the same conditions that are given in Problem 5.

7. Use the series solutions from Problems 5 and 6 to obtain corresponding series solutions for the homology invariants u and v.

8. Calculate a value for the free-fall time for an object on the surface of the sun to arrive at the center of the sun.

9. Show that the results of equation (2.3.8) are indeed correct. State clearly all assumptions you make during your derivation.

References and Supplemental Reading

1. Chandrasekhar S.: *An Introduction to the Study of Stellar Structure* Dover, New York, 1957 p. 77.

For those who are interested in a further discussion of the integral theorems, some excellent articles are:

Chandrasekhar, S.: *An Integral Theorem on the Equilibrium of a Star*
Ap. J. 87, 1938, pp. 535 - 552;

________________ *The Opacity in the Interior of a Star* Ap. J. 86, 1937, pp. 78 - 83;
________________ *The Pressure in the Interior of a Star*_ Ap. J. 85, 1937, pp. 372 - 379;
________________ *The Pressure in the Interior of a Star.*_ Mon. Not. R. astr. Soc. 96, 1936, pp. 644 - 647.

Milne, E.A.: *The Pressure in the Interior of a Star* Mon. Not. R. astr. Soc. 96, 1936, pp. 179 - 184.

For a complete discussion of polytropes and isothermal spheres see any of these:

Chandrasekhar S.: *An Introduction to the Study of Stellar Structure* Dover, New York, 1957, chap. 4, p. 84.

Eddington, A.S.: *The Internal Constitution of the Stars* Dover, New York, 1959, chap. 4, p. 79.

Cox, J. P., and Giuli, R. T.: *Principles of Stellar Structure* Gordon & Breach, New York, 1968, Chap. 12, p. 257.

An interesting example of the use of polytropes to explore the more complicated phenomenon of rotation can be found in

Limber, D. N., and Roberts, P.H. : *On Highly Rotating Polytropes V*, Ap. J. 141, 1965, pp.1439-1442.

Geroyannis, V.S., and Valvi, F. N. : *Numerical Implementation of a Perturbation Theory Up to Third Order for Rotating Polytropic Stars: Parameters Under Differential Rotation*, Ap.J. 312, 1987, pp. 219-226.

A brief but useful account of the physical nature of polytropes may be found in

Clayton, D. D.: *Principles of Stellar Evolution and Nucleosynthesis*, McGraw-Hill, New York, 1968, pp. 155-158.