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Flow of Energy through the Star and Construction of Stellar Models

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That the central temperatures of stars are higher than their surface temperatures can no longer be in doubt. The laws of thermodynamics thus ensure that energy will flow from the center of stars to their surfaces. The physical processes that accomplish this will basically establish the temperature gradient within the star. This is the remaining relationship required for us to link the interior structure with that of the surface. The temperature gradient and along with the conservation laws of mass and energy provide the three independent relationships necessary to relate the three state variables to the values they must have at the boundaries of the star.

Energy can move through a medium by essentially three ways, and they can each be characterized by the gas particles which carry the energy and the forces which resist these efforts. These mechanisms are *radiative transfer, convective tran*sport, and *conductive transfer*, of energy. The efficiency of these processes is determined primarily by the amount of energy that can be carried by the particles, their number and their speed. These variables set an upper limit to the rate of energy transport. In addition, the "opacity" of the material to the motion of the energy-carrying particles will also affect the efficiency. In the case of radiation, we have characterized this opacity by a collision cross section and the density. Another way to visualize this is via the notion of a mean free path. This is just the average distance between collisions experienced by the particles. In undergoing a collision, the particle will give up some of its energy thereby losing its efficiency as a transporter of energy. We will see that, in general, there are large differences in the mean free paths for the particles that carry energy by these three mechanisms, and so one mechanism will usually dominate in the transfer of energy.

Before we can describe the radiative flow of energy, we must understand how the presence of matter impedes that flow. Thus we shall begin our discussion of the transport of energy by determining how the local radiative opacity depends on local values of the state variables. From the assumption of strict thermodynamic equilibrium (STE) we know that any impediment to the flow can be described in terms of a parameter that depends only on the temperature. However, to calculate that parameter, we have to investigate the detailed dependence of the opacity on frequency.

4.1 The Ionization, Abundances, and Opacity of Stellar Material

We have now described the manner in which nuclear energy is produced in most stars, but before we can turn to the methods by which it flows out of the star, we must quantitatively discuss the processes which impede that flow. Each constituent of the gas will interact with the photons of the radiation field in a way that is characterized by the unique state of that particle. Thus, the type of atom, its state of ionization, and excitation will determine which photons it can absorb and emit. It is the combination of all the atoms, acting in consort that produces the opacity of the gas. The details which make up this combination can be extremely complicated. However, several of the assumptions we have made, and justified, will make the task easier and certainly the principles involved can be demonstrated by a few examples.

a Ionization and the Mean Molecular Weight

Our first task is to ascertain how many of the different kinds of particles that make up the gas are present. To answer this question, we need to know not only the chemical makeup of the gas, but also the state of ionization of the atoms.

We have already established that the temperatures encountered in the stellar interior are very high, so we might expect that most of the atoms will be fully ionized. While this is not exactly true, we will assume that it is the case. A more precise treatment of this problem will be given later when we consider the state of the gas in the stellar atmosphere where, the characteristic temperature is measured in thousands of degrees as opposed to the millions of degrees encountered in the interior.

We will find it convenient to divide the composition of the stellar material into three categories.

$$\mathbf{X} = \text{mass fraction of gas which is hydrogen}$$
$$\mathbf{Y} = \text{mass fraction of gas which is helium}$$
$$\mathbf{Z} = \text{mass fraction of gas which is everything else} \tag{4.1.1}$$

It is common in astronomy to refer to everything which is not hydrogen or helium as "metals". For complete ionization, the number of particles contributed to the gas *per element* is just

$$n_i = \frac{Z_i + 1}{A_i} \tag{4.1.2}$$

where Z_i is the atomic number and A_i is the atomic weight of the element. Thus, the number of particles contributed by hydrogen will just be twice the hydrogen abundance, and for helium, three-fourths times the helium abundance. In general,

$$n_i \approx \frac{Z_i + 1}{2Z_i} \tag{4.1.3}$$

The limit of equation (4.1.3) for the heavy metals is ½. However, even at 10^7 K the inner shells of the heavy metals will not be completely ionized and so ½ will be an overestimate of the contribution to the particle number. This error is somewhat compensated by the 1 in the numerator of equation (4.1.3) for the light elements where it provides an underestimate of the particle contribution. Thus we take the total number of particles contributed by the metals to be ½ **Z**. The total number of particles in the gas from all sources is then

$$N = \frac{\rho}{m_h}\sum_i n_i x_i = \frac{\rho}{m_h}\left(2\mathbf{X} + \tfrac{3}{4}\mathbf{Y} + \tfrac{1}{2}\mathbf{Z}\right) \tag{4.1.4}$$

Since everything in the star is classed as either hydrogen, helium, or metals, $\mathbf{X} + \mathbf{Y} + \mathbf{Z} = 1$ and we may eliminate the metal abundance from our count of the total number of particles, to get

$$N = \frac{1}{2}\left(\frac{\rho}{m_h}\right)\left(3\mathbf{X} + \frac{1}{2}\mathbf{Y} + 1\right) \tag{4.1.5}$$

Throughout the book we have introduced the symbol m as the mean molecular weight of the gas without ever providing a clear definition for the quantity. It is clearly time to do so, and we can do it easily given the ideal-gas law and our expression for the total number of particles N. Remember

$$P = NkT = \frac{\rho kT}{\mu m_h} \tag{4.1.6}$$

The mean molecular weight μ must be defined so that this is a correct expression. Thus,

$$\mu = \frac{2}{1 + 3X + \frac{1}{2}Y} \tag{4.1.7}$$

b Opacity

In general, a photon can interact with atoms in three basic ways which result in the photon being absorbed:

1. Bound-bound absorption (atomic line absorption)
2. Bound-free absorption (photoionization)
3. Free-free absorption (bremsstrahlung)

For us to calculate the impeding effect of these processes on the flow of radiation, we must calculate the cross section for the processes to occur for each type of particle in the gas. For a particular type of atom, this parameter is known as the *atomic absorption coefficient* α_n. The atomic absorption coefficient is then weighted by the abundance of the particle in order to produce the *mass* absorption coefficient κ_ν, which is the opacity per *gram* of stellar material at a particular frequency ν. Since we have assumed that the gas and photons are in STE, we can average the opacity coefficient over frequency to determine a mean opacity coefficient $\overline{\kappa}$, which will represent the average effect on the diffusion of energy through the star of the material itself. However, to calculate this mean, we must calculate κ_ν itself. As an example of how this is done, we shall consider the atomic absorption coefficient of hydrogen and "hydrogen-like" elements.

Classical View of Absorption Imagine a classical electromagnetic wave encountering an atom. The time-varying electromagnetic field will cause the electron to be accelerated so that it oscillates at the same frequency as the wave. However, just as an accelerated charge radiates energy, to accelerate a charge requires energy, and in this instance it is the energy of the electromagnetic wave. If the electron is bound in an atom, the energy may just be sufficient to raise the electron to a higher orbit, and we say that a *bound-bound transition* has taken place. If the energy is sufficient to remove the electron and ionize the atom, we say that a *bound-free transition*, or *photoionization*, has taken place. Finally, if a free electron is passing an ion in an unbound orbit, it is possible for the electron to absorb the energy from an electromagnetic wave that happens to encounter this system. Energy and momentum are conserved among the two particles and the photon with the result that the electron is moved to a different unbound orbit of higher energy relative to the ion. This is known as a *free-free absorption*.

Quantum Mechanical View of Absorption In quantum mechanics the classical view of a finite cross section of an atom for electromagnetic radiation is replaced by the notion of a transition probability. That is, one calculates the probability that an electron will make a transition from some initial state to another state while in the presence of a photon. One calculates this probability in terms of the wave functions of the two states, and it usually involves a numerical integration of the wave functions over all space. Instead of becoming involved in the detail, we shall obtain a qualitative feeling for the behavior of this transition probability.

Within the framework of quantum mechanics, the probability that an electron in an atom will have a specific radial coordinate is

$$\langle\vec{r}\rangle = \int_V \Psi_i \vec{r} \Psi_i^* \, dV \tag{4.1.8}$$

where i denotes the particular quantum state of the electron (that is, n,j,l) and Ψ_i is the wave function for that state. In classical physics, the dipole moment $\vec{P}$ of a charge configuration is

$$\vec{P} = \int_v \vec{r} \rho_c(\vec{r}) \, dV \tag{4.1.9}$$

where ρ_c is the charge density. The quantum mechanical analog is

$$\langle\vec{P}_{ij}\rangle = e \int_v \Psi_i \vec{r} \Psi_j \, dV \tag{4.1.10}$$

where i denotes the initial state and j the final state.

Now, within the context of classical physics, the energy absorbed or radiated per unit time by a classical oscillating dipole is proportional to $\vec{P} \bullet \vec{P}$, and this result carries over to quantum mechanics. This classical power is

$$\mathbf{P} \propto \nu^4 P^2 \tag{4.1.11}$$

Since the absorbed power **P** is just the energy absorbed per second, the number of photons of energy hν that are absorbed each second is

$$N_p \propto \nu^3 P^2 \tag{4.1.12}$$

However, the number of photons absorbed per second will be proportional to the probability that one photon will be absorbed, which is proportional to the collision cross section. Thus, we can expect the atomic cross section to have a dependence on frequency given by

$$\alpha_\nu = \frac{\text{const}}{\nu^3 P^2} \tag{4.1.13}$$

In general, we can expect an atomic absorption coefficient to display the ν^{-3} dependence while the constant of proportionality can be obtained by finding the dipole moment from equation (4.1.10). The result for the bound-free absorption of hydrogen and hydrogenlike atoms is

$$\begin{aligned}
\alpha_\nu^{\text{b-f}}(i, n) &= \frac{64\pi^4 Z_i^4 m_e e^{10}}{3\sqrt{3}h^6 c} \frac{1}{n^5} S_{ni}^4 g_\nu^{\text{b-f}}(i, n)] \left(\frac{1}{\nu^3}\right) \\
\alpha_\nu^{\text{b-f}}(i, n) &= (2.815 \times 10^{29}) \frac{Z_i^4 S_{ni}^4 g_\nu(i, n)}{n^5 \nu^3} \ \text{cm}^2
\end{aligned} \tag{4.1.14}$$

where

i = state of ionization

n = principal quantum number of electron

Z_i = atomic number

S_{ni} = screening factor resulting from interior electrons

$g_\nu(i, n)$ = gaunt factor ≈ 1

(4.1.15)

A similar expression can be developed for the free-free transitions of hydrogen-like atoms:

$$\alpha_n^{ff}(i,p)dn_e(p) = \frac{4\pi Z_i^2 e^6 S_{fi}^2}{3\sqrt{3}hcm_e^2 v(p)} g_n^{ff}(1/n^3)dn_e(p) \tag{4.1.16}$$

Here, the atomic absorption coefficient depends on the momentum of the "colliding" electron. If one assumes that the momentum distribution can be obtained from Maxwell-Boltzmann statistics, then the atomic absorption coefficient for free-free transitions can be summed over all the colliding electrons and combined with that of the bound-free transitions to give a *mass* absorption coefficient for hydrogen that looks like

$$\kappa_\nu(\text{hydrogen}) = \frac{32\pi^2 e^6 R e^{-\chi_0/(kT)}}{3\sqrt{3}h^3 c m_h \nu^3}\left(\sum_{\nu_n<\nu}^{\infty} \frac{e^{-\epsilon_n/(kT)}}{n^3}\bar{g}_n + \frac{\bar{g}_{ff}kT}{2\chi_0}\right)$$

where

χ_0 = ionization potential of hydrogen
R = gas constant
ϵ_n = excitation energy of ν_nth energy level
$\bar{g}_n, \bar{g}_{ff}$ = gaunt factors averaged over frequency

(4.1.17)

The summation in equation (4.1.17) is to be carried out over all n such that $\nu_n < \nu$. That is, all series that are less energetic than the frequency ν can contribute to the absorption coefficient. For us to use these results, they must be carried out for each element and combined, weighted by their relative abundances. This yields a frequency-dependent opacity per gram κ_ν which can be further averaged over frequency to obtain the appropriate average effect of the material in impeding the flow of photons through matter. However, to describe the mean flow of radiation through the star, we want an estimate of the transparency of transmissivity of the material. This is clearly proportional to the inverse of the opacity. Hence we desire a reciprocal mean opacity. This frequency-averaged reciprocal mean is known as the Rosseland mean and is defined as follows:

$$\frac{1}{\bar{\kappa}_\nu} \equiv \frac{\int_0^\infty (1/\kappa_\nu)\,\partial B_\nu(T)/\partial T\,d\nu}{\int_0^\infty [\partial B_\nu(T)/\partial T]\,d\nu} \tag{4.1.18}$$

Here, $B_\nu(T)$ is the Planck function, which is the statistical equilibrium distribution function for a photon gas in STE which we developed in Chapter 1 [equation (1.1.24)]. That such a mean should exist is plausible, since we are concerned with the flow of energy through the star, and as long as we assume that the gas and photons are in STE, we know how that energy must be distributed with wavelength. Thus, it would not be necessary to follow the detailed flow of photons in frequency space since we already have that information. That there should exist an average value of the opacity for that frequency distribution is guaranteed by the mean value theorem of calculus. That the mean absorption coefficient should have the form given by equation (4.1.18) will be shown after we have developed a more complete theory of radiative transfer (see Section 10.4).

Approximate Opacity Formulas Although the generation of the mean opacity coefficientk is essentially a numerical undertaking, the result is always a function of the state variables P, T, ρ, and μ. Before the advent of the monumental studies of Arthur Cox and others which produced numerical tables of opacities, much useful work in stellar interiors was done by means of expressions which give the approximate behavior of the opacity in terms of the state variables. The interest in these formulas is more than historical because they provide a method for predicting the behavior of the opacity in stars and a basis for understanding its relationship to the other state variables. If one is constructing a model of the interior of a star, such approximation formulas enable one to answer the question so central to any numerical calculation: Are these results reasonable? In general, these formulas all have the form

$$\begin{array}{lll} & \bar{\kappa} = \bar{\kappa}_0 \rho^n T^{-s} & \\ (a)\ n = 1 & s = 3.5 & \text{Kramers' law} \\ (b)\ n = 0.75 & s = 3.5 & \text{Schwarzschild's opacity} \\ (c)\ n = 0 & s = 0 & \text{electron scattering} \end{array} \qquad (4.1.19)$$

where $\overline{\kappa}_0$ depends on the chemical composition μ . Kramer's opacity is a particularly good representation of the opacity when it is dominated by free-free absorption, while the Schwarzschild opacity yields somewhat better results if bound-free opacity makes an important contribution. The last example of electron scattering requires some further explanation since it is not strictly a source of absorption.

Electron Scattering The scattering of photons at the energies encountered in the stellar interior is a fully conservative process in that the energy of the photon can be considered to be unchanged. However, its direction is changed, resulting in the photon describing a random walk through the star. This immensely lengthens the path taken by the photon and therefore increases its "stay" in the star. The longer the photon resides in the star, the greater its path, and the greater are its chances of being absorbed by an encounter with an atom. Thus, electron scattering, while not involved directly in the absorption of photons, does significantly contribute to the opacity of the gas. The photon flow is impeded by electron scattering, first, by redirecting the photon flow and, second, by lengthening the path and increasing the photon's chances of absorption.

As long as $h\nu \ll m_e c^2$, the electron will exhibit little or no recoil as a result of its collision with a photon and the photon energy will be unchanged. This case is called Thomson scattering and we can use the classical theory of electromagnetism to estimate its cross section. The energy radiated or absorbed per unit time by an oscillating free electron is

$$\frac{dE}{dt} = \frac{2e^4E_0^2}{3m_e^2c^3}\text{Sin}^2\,\omega t \tag{4.1.20}$$

However, the power in an electromagnetic wave is given by the Poynting vector $\vec{S} = [c/(4\pi)](\vec{E}\times\vec{H})$. In vacuum, $\vec{E}\perp\vec{H}$, and in Gaussian units $|E|=|H|$. Therefore, the magnitude of the Poynting vector is

$$S = \frac{c}{4\pi}E_0^2\,\text{Sin}^2\,\omega t \tag{4.1.21}$$

The ratio of the power absorbed from the wave to the power in the wave is (dE/dt)/*S* and is the definition of a cross section. Thus, we can write the classical cross section for electron scattering as

$$\sigma_0 = \frac{8\pi}{3}\left(\frac{e^2}{m_ec^2}\right)^2 = 6.652\times10^{-25}\,\text{cm}^2 \tag{4.1.22}$$

The quantity $e^2/(m_ec^2)$ is known as the *classical radius* of the electron and is roughly that radius for which the field energy of the electron is equal to its rest energy. The square of that radius yields a geometric cross section which is 1.5 times the classical cross section. This cross section is also known as the *Thomson cross section.*

Note that the Thomson cross section is not a function of frequency, which makes it particularly easy to incorporate in an expression for opacity. The symbol s_e usually denotes the electron scattering coefficient per *gram* of stellar material, so that

$$\sigma_e = \frac{\sigma_0 n_e}{\rho} \tag{4.1.23}$$

However, in the limit where the total number of particles contributed to the particle density by metals is ½**Z** and most of those are electrons, we get the electron density to be

$$n_e = \frac{\rho}{m_h}\left\langle\frac{Z}{A}\right\rangle \approx \frac{1}{2}\left(\frac{\rho}{m_h}\right)(1+X) \tag{4.1.24}$$

which yields an electron scattering cross section *per gram* of

$$\sigma_e \approx 0.2004(1+X)\,\text{cm}^2/\text{g} \le 0.4\,\text{cm}^2/\text{g} \tag{4.1.25}$$

Thus, there is a limit to the scattering coefficient per gram of stellar material of something less than 0.5 cm^2/g.

We have now shown how the energy generation rate and opacity per gram of stellar material can be related to the state variables P, T, ρ, and μ. You must not assume that this discussion is complete in every detail. The detailed calculation of these functions is extremely complex and has entertained some of the best minds of the twentieth century. What we have seen is some of the major physical principles which affect the outcome of such efforts. For the details of the modern values of these functions, you should consult the current literature as refinements continue. Nevertheless, from now on, we may assume that we have functions of the form

$$\epsilon = \epsilon(P, T, \rho, \mu) \qquad \kappa = \kappa(P, T, \rho, \mu) \tag{4.1.26}$$

at our disposal. Now we turn to the problem of describing the flow of energy through the star.

4.2 Radiative Transport and the Radiative Temperature Gradient

Although all forms of energy transport may be present at any given place in a star, we will see that their relative efficiency is such that generally only one form will be important for describing the flow of energy. The transport of energy by radiation is essentially the radiative diffusion of photons through the stellar material. It is the opacity of the material that opposes this flow. To establish the interplay between thermodynamics and radiative opacity, we assume that all the energy is flowing by this process.

a Radiative Equilibrium

Since we are assuming that all the energy is flowing outward by means of radiative diffusion, the entire energy produced by the star within a sphere of radius r can be characterized by a local luminosity L(r) which is entirely made up of photons. When this is the case, we may describe this flow of photons locally by defining the radiative flux as

$$F(r) = L(r)/4\pi r^2 \tag{4.2.1}$$

When these conditions prevail, the entire flow of energy is carried by photons and the star is said to be in *radiative equilibrium.*

b Thermodynamic Equilibrium and Net Flux

In Chapter 1 we developed an elegant formalism to describe the flow of particles through space. In a later chapter we shall use this to produce an extremely general equation of radiative transfer which describes the flow in momentum space as well as physical space. But at this point, we are dealing with a gas in STE, and that fixes many properties of the gas. For example, we know that the phase density f that appears in the Boltzmann transport equation will be the Planck function since we have shown that to be the equilibrium distribution function for photons in STE. We also know that while there is a net flow of photons, the energy involved in that flow must be small compared to the local energy density; otherwise, the photon gas could not be considered in equilibrium.

Another way of visualizing this is to observe that any system said to be in thermodynamic equilibrium cannot have temperature gradients. If it did, there would be a flow of energy driven by the temperature gradient. In a star we must have such a flow, or the star will not shine. What is important is the relative size of the temperature radiant through some volume for which the system is to be considered in equilibrium. In the case of the sun, this typical length would be the distance a photon travels before it encounters an atom. From the opacity calculations of Chapter 3 and our knowledge of the conditions within the sun, we would calculate that the mean free path for a photon in the center of the sun is less than a centimeter. Thus, as a measure of the extent to which STE is met in the sun, let us calculate

$$\left(\frac{\nabla T}{T}\right) l \approx \left(\frac{T_c/R}{T_c}\right) l \approx \frac{l}{R} \approx 10^{-11} \tag{4.2.2}$$

In other words, the change in the local temperature over a scale length appropriate for the photon gas is about 1 part in 10^{11}. There are few gaseous structures in the universe where the conditions for STE are met better than this. Small as this relative temperature gradient is, it drives the luminous flux of the sun, and so we must estimate its dependence on the state variables.

c Photon Transport and the Radiative Gradient

Since we know so much about the nature of the photons in the star, we need not resort to the basic Boltzmann transport equation in order to describe how photons flow. Instead, consider the Euler-Lagrange equations of hydrodynamic flow. Since they were derived under fairly general conditions, they should be adequate to describe the flow of photons. Equation (1.2.27) provides a reasonably simple description of this process. But we are interested in a steady-state description, so all explicit time dependence in that equation must vanish. Thus, equation (1.2.27) becomes

$$(\vec{u} \cdot \nabla)\vec{u} = -\nabla\Phi - \left(\frac{1}{\rho}\right)\nabla P \tag{4.2.3}$$

However, in deriving equation (1.2.27), we averaged the local particle phase density over velocity space. For photons traveling at the velocity of light, this does not make much sense. Instead, the moment generation which led to the Euler-Lagrange equations of hydrodynamic flow should be carried out over momentum space, or photon frequency. Since this expression is for photons, P is the local radiation pressure due to photons and $\vec{u}$ is the mean flow velocity, or diffusion velocity of those photons. However, in equation (1.2.27), p is the local mass density. For photons, this translates to the local energy density. In addition, the influence of gravity on the photons throughout the star can be estimated by the gravitational red shift that photons will experience which is

$$\frac{\Delta h\nu}{h\nu} \approx \frac{GM}{Rc^2} \approx 10^{-6} \tag{4.2.4}$$

Since the change in the photon energy resulting from moving through the gravitational potential is about 1 part in a million, we may safely neglect the influence of $\nabla\Phi$. In spherical coordinates, all spatial operators in equation (4.2.3) simply become derivatives with respect to the radial coordinate, so that equation (4.2.3) becomes

$$\vec{u}_r \cdot \left(\frac{d\vec{u}_r}{dr}\right)\hat{r} = -\left(\frac{1}{\rho_e}\right)\nabla P_r \tag{4.2.5}$$

where

$$\rho_e = \langle h\nu \rangle \int_0^\infty f(p)\,\frac{d\vec{p}}{c^2}$$

$$\vec{u}_r = \left(\frac{\langle h\nu \rangle}{c^2}\right)^{-1} \frac{\int_0^\infty \vec{p} f(p)\,d\vec{p}}{\int_0^\infty f(p)\,d\vec{p}} \tag{4.2.6}$$

Here <hν> is the average photon energy, and $d\vec{p}$ indicates integration over all momentum coordinates which, in the absence of a strong potential gradient, can be represented by the differential spherical momentum volume $4\pi p^2 dp$. We can then write equation (4.2.5) as

$$\nabla P_r = -\left(\int_0^\infty \vec{p} f(p)\,d\vec{p}\right) \cdot \left[\frac{d}{dr}\int_0^\infty \frac{\vec{p} f(p)\,d\vec{p}}{\rho_e}\right] \tag{4.2.7}$$

The first term on the right-hand side represents the net flow of momentum and can be related to the flow of radiant energy by

$$\int_0^\infty \vec{p} f(p)\,d\vec{p} = \frac{1}{c}\int_0^\infty h\nu f(p)\hat{r}(4\pi p^2)\,dp = \frac{\vec{F}(r)}{c} \tag{4.2.8}$$

and equation (4.2.7) becomes

$$\nabla P_r = \frac{-\vec{F}(r)}{c} \cdot \left\{\frac{d}{dr}\left[\frac{F(r)}{\rho_e c}\right]\right\} \tag{4.2.9}$$

The quantity $[F(r)/c\rho_e]$ is the fraction of photons which are participating in the net flow of energy. Thus the radial derivative represents the change in the fraction with r. The only reason for this fraction to change is the interaction of the flowing photons with matter. If we define the volume absorption coefficient α_v, to be the "collision" cross section per unit volume, then the probability per unit length that a photon will be absorbed in passing through that volume is just α_v. However, the probability that one photon will be absorbed per unit length is equal to the fraction of *n* photons that will be absorbed in that same unit length. Thus, the second term on the right hand side of equation (4.2.9) becomes

$$\frac{d}{dr}\left[\frac{F(r)/c}{\rho_e}\right] = \bar{\alpha}_v = \bar{\kappa}\rho \tag{4.2.10}$$

The radiation pressure gradient is now

$$\nabla P_r = -\frac{\bar{\kappa}\rho F(r)}{c} = \frac{-\bar{\kappa}\rho L(r)}{4\pi c r^2} \tag{4.2.11}$$

But in STE the radiation pressure depends on only a single parameter and is given by

$$P_r = \frac{aT^4}{3} \tag{4.2.12}$$

This implies we can write the radiation pressure gradient in terms of the temperature as

$$\frac{dP_r}{dr} = \frac{4}{3}aT^3\frac{dT}{dr} \tag{4.2.13}$$

Equating this to the magnitude of the radiation pressure gradient from equation (4.2.11), we finally obtain an expression for the radiative temperature gradient:

$$\frac{dT}{dr} = -\frac{3\bar{\kappa}\rho L(r)}{16\pi a c r^2 T^3} \tag{4.2.14}$$

This relationship specifies how the temperature must change if the energy is being carried by radiative diffusion and the specification is made in terms of the state variables and parameters that we have already determined characterize the problem.

d Conservation of Energy and the Luminosity

With the advent of the radiant flux F(r), we have introduced a new variable into the problem. Relating the flux to the total luminosity [equation (4.2.1)] only transfers the source of the problem to the luminosity L(r). That such a parameter is important should surprise no one, for the luminosity of a star is perhaps its most obvious characteristic. However, it is only with the transport of energy that we are faced with the internal energy, stored or produced, arriving at the surface and leaking into space. As far as the structure of the star is concerned, this is a "second-order" effect. It is only a small part of the internal energy that is lost during a dynamical time interval. However, for the proper understanding of the star as an object in steady state, it is a central condition which must be met, for in steady state the energy lost must be matched by the energy produced.

Fortunately, we have an additional fundamental constraint that must be met by any physical system which we have not yet imposed - the conservation of energy. This is completely analogous to the conservation of mass which we invoked in Chapter 2 [equations (2.1.7) and (2.1.8)] only now it is the total energy interior to r which must pass through r per unit time is called L(r). Thus,

$$L(r) = \int_0^r 4\pi r^2 \rho \epsilon \, dr \tag{4.2.15}$$

The corresponding differential form is

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho \epsilon \tag{4.2.16}$$

4.3 Convective Energy Transport

Our approach to the transport of energy by convection will be somewhat different from that for radiation. For radiation, we knew how much energy there was to carry $-[L(r)/4\pi r^2]$, and we set about finding the temperature gradient required to carry it. For convection, we will anticipate the answer by calculating the amount of energy that a super-adiabatic temperature gradient will carry. For a wide range of parameters thought to prevail in the stellar interior, we shall discover that the adiabatic gradient is adequate to carry all the required energy. But first we must determine the adiabatic temperature gradient.

a Adiabatic Temperature Gradient

In Chapter 2 [equation (2.4.6)] we defined polytropic change in terms of a specific heat-like quantity C which is equal to the change in heat with respect to temperature. For an adiabatic change, the gas does no work on the surrounding medium, so that C = 0. The polytropic γ' as defined by equation (2.4.8) is

$$\gamma' = \gamma = \frac{C_P}{C_V} \tag{4.3.1}$$

Using equation (2.4.5) and the ideal-gas law, we have

$$\gamma = 1 + \frac{R}{C_V} = 1 + \frac{Nk}{\frac{3}{2}Nk} = \frac{5}{3} \tag{4.3.2}$$

or

$$n = \tfrac{3}{2} \tag{4.3.3}$$

where *n* is the appropriate polytropic index for an adiabatic gas.

Now the polytropic equation of state [equation (2.4.1)] and the ideal-gas law guarantee that

$$\begin{aligned} P &= k\rho^{(n+1)/n} \\ T &= \left(\frac{\mu m_h k}{k}\right)\rho^{1/n} \end{aligned} \tag{4.3.4}$$

Forming the logarithmic derivative of P and T with respect to ρ we get

$$\begin{aligned} \frac{1}{P}\frac{dP}{d\rho} &= \frac{n+1}{n}\rho^{-1} \\ \frac{1}{T}\frac{dT}{d\rho} &= \frac{1}{n}\rho^{-1} \end{aligned} \tag{4.3.5}$$

Dividing these two equations yields

$$\frac{T}{P}\frac{dP}{dT} = n + 1 \tag{4.3.6}$$

which is known as the *polytropic temperature gradient* and for an adiabatic gas is just

$$\frac{d\ln P}{d\ln T} = 2.5 \tag{4.3.7}$$

b Energy Carried by Convection

Imagine a small element of matter rising as a result of being somewhat hotter than its surroundings (see Figure 4.1). We can express the temperature difference between the gas element and its surroundings in terms of the external temperature gradient and the internal temperature gradient experienced by the small element as it rises. We assume that the element is behaving adiabatically, and so this internal gradient is the adiabatic gradient and the temperature difference is

$$\delta T = \left.\frac{dT}{dr}\right|_{\text{ad}}\delta r - \left.\frac{dT}{dr}\right|_{\text{ext}}\delta r \equiv \Delta\nabla T\,\delta r \tag{4.3.8}$$

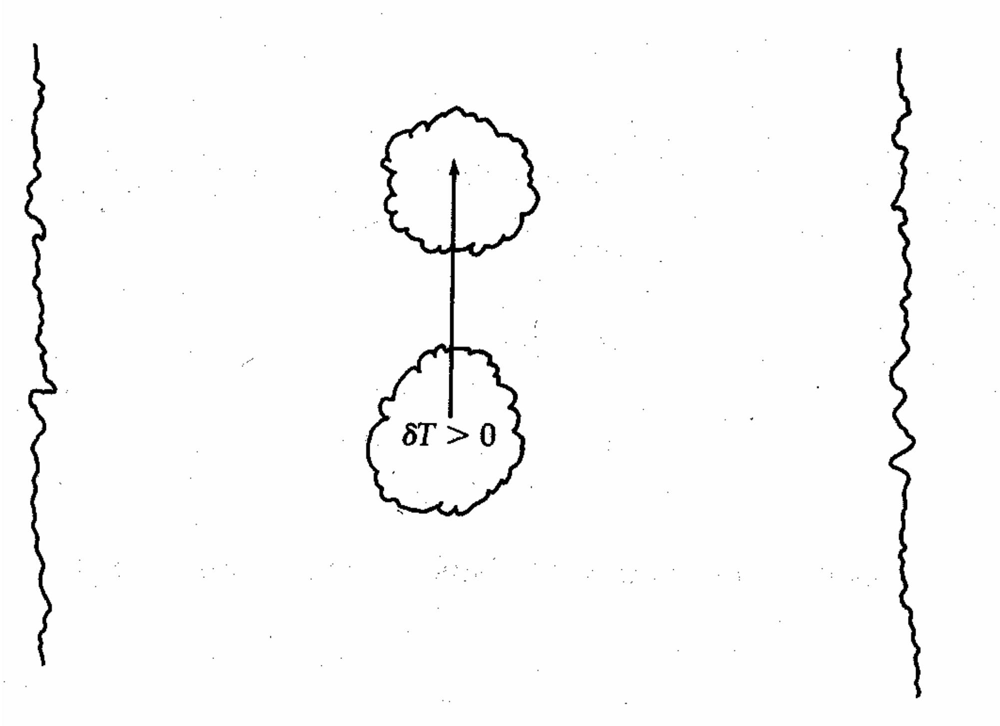

Figure 4.1 shows a schematic representation of a convective element of gas responding to a small temperature difference.

The flux of energy carried by this small convective element will be

$$F_{\text{conv}} = (\Delta\nabla T\,\delta r)\rho C_P v \tag{4.3.9}$$

where v is the average velocity of the convective element which we must estimate. The buoyant force experienced by the convective element will be determined solely by the density difference resulting from its slightly elevated temperature and is

$$F_b = \frac{Gm(r)}{r^2}\,\delta\rho \tag{4.3.10}$$

Since the convective element rises adiabatically, pressure equilibrium will always be maintained during its ascent. Thus we can relate the variation in density to the variation in temperature by setting the variation of the ideal-gas law to zero.

$$\delta P = \frac{kT}{\mu m_h}\,\delta\rho + \frac{k\rho}{\mu m_h}\,\delta T = 0 \tag{4.3.11}$$

We may use this and equation (4.3.8) to obtain the average buoyancy acting on the convective element. Initially, the buoyancy force is zero since the gradient difference does not produce a significant force until the element has traveled some distance. Thus, we take the average force to be one-half the maximum force, and we get

$$\langle F_b \rangle = -\frac{1}{2}\frac{GM(r)}{r^2}\frac{\rho}{T}\Delta\nabla T\,\delta r \tag{4.3.12}$$

Now the buoyancy force will continuously accelerate the convective element, giving it a kinetic energy of (½)ρv^2 which we can use to get an estimate of the convective velocity v. Thus,

$$\langle F_b \rangle\,\delta r = \tfrac{1}{2}\rho v^2 = \left[\frac{1}{2}\frac{GM(r)}{r^2}\frac{\rho}{T}(\Delta\nabla T\,\delta r)\right]\delta r \tag{4.3.13}$$

which yields a convective velocity of

$$v_{\mathrm{conv}} \approx \left[\frac{GM(r)}{Tr^2}\right]^{1/2}(\Delta\nabla T)^{1/2}\,\delta r \tag{4.3.14}$$

We define

$$l = 2\,\delta r \tag{4.3.15}$$

This quantity *l* is known as the mixing length and is largely a free parameter of this theory of convection from which it takes its name. Typically it is taken to be of the order of a pressure scale height, and fortunately for the theory of stellar interiors, the results are not too sensitive to its exact value. In terms of the mixing length, the convective flux becomes

$$F_{\mathrm{conv}} = C_P\rho\left[\frac{GM(r)}{Tr^2}\right]^{1/2}(\Delta\nabla T)^{3/2}\frac{l^2}{4} \tag{4.3.16}$$

Now all that remains is to estimate the difference in temperature gradients necessary to transport the energy of the star. We will require that the convection carry all the internal energy flowing through the star, so that

$$F_{\mathrm{conv}} = \frac{L(r)}{4\pi r^2} \tag{4.3.17}$$

which yields the gradient difference of

$$\Delta\nabla T = \left[\frac{L^2(r)T}{C_P^2\rho^2 GM(r)\pi^2 l^4 r^2}\right]^{1/3} \tag{4.3.18}$$

To arrive at some estimate of the significance of this result, let us compare it to the adiabatic gradient. We use the adiabatic temperature gradient in equation (4.3.7), hydrostatic equilibrium [equation (1.2.28)], and the ideal-gas law to get

$$(\nabla T)_{\mathrm{ad}} = \frac{dT}{dP}\frac{dP}{dr} = -\frac{\mu m_h G M(r)}{2.5 k r^2} \tag{4.3.19}$$

Dividing equation (4.3.18) by the adiabatic gradient we get

$$\frac{\Delta \nabla T}{(\nabla T)_{\mathrm{ad}}} = \frac{5}{2}\left(\frac{L(r)}{C_P \pi \rho}\right)^{2/3} T^{1/3}\left[\frac{r}{GM(r)l}\right]^{4/3}\left(\frac{\mu m_h}{k}\right)^{-1} \tag{4.3.20}$$

For the sun, there is some evidence that a mixing length of about one-tenth of a solar radius is not implausible. Picking other values for the sun and trying to maximize equation (4.3.20), we have the following selection:

$$\begin{array}{ll} r = \dfrac{R}{10} \approx l & M(r) = M_{\odot} \\ T = 10^7\ \mathrm{K} & \mu = 0.7 \\ \rho \approx 1\ \mathrm{g/cm^3} & L = L_{\odot} \\ C_P = 3.5 & \dfrac{\Delta \nabla T}{(\nabla T)_{\mathrm{ad}}} \approx 4 \times 10^{-3} \end{array} \tag{4.3.21}$$

Thus, it would seem that the convective gradient will lie within a few tenths of a percent of the adiabatic gradient. This is the source of the statement in Chapter 2 that a polytrope of index 3/2 represents convective stars quite well. Indeed, convection is so efficient that the adiabatic gradient will almost always suffice to describe convective stellar interiors. This is fortunate since the mixing length theory we have discussed here is admittedly rather crude. Unfortunately, this efficiency does not carry over into stellar atmospheres because the convective zones are bounded by the surface of the star, dropping the mixing lengths to numbers comparable to the photon mean free path so that radiation competes effectively with convection regardless of the temperature gradient. For stellar interiors, the photon mean free path is measured in centimeters and the mixing length in fractions of a stellar radius. Thus convection, when established, will always be able to carry the stellar luminosity with a temperature gradient close to the adiabatic gradient.

4.4 Energy Transport by Conduction

a Mean Free Path

Consider a simple monatomic gas where the kinetic energy per particle is 3kT/2 so that the speed is

$$v = \left(\frac{3kT}{m}\right)^{1/2} \tag{4.4.1}$$

We will let the collisional cross section be just the geometric cross section, so that

$$\sigma = \pi(r_1 + r_2)^2 \tag{4.4.2}$$

where r_1 and r_2 are the radii of the two species of colliding particles. As we did with nuclear reaction rates, we get the collision frequency from the effective volume swept per unit time σv multiplied by the number density ρ/m. The time between collisions is just the reciprocal of the collision frequency, so that the distance traveled between collisions is

$$l = v\left(\frac{\sigma v \rho}{m}\right)^{-1} = \frac{m}{\rho\sigma} \tag{4.4.3}$$

and is known as the *mean free path* for collisions.

b Heat Flow

The thermodynamic theory of heat says that the heat flux through a given area is proportional to the temperature gradient so that

$$F_{\text{cond}} = -\mathrm{K}\nabla T \tag{4.4.4}$$

where Eddington[1] gives the conductivity **K** as

$$\mathrm{K} = \frac{C_V \rho l v}{3} = \frac{C_V(kTm/3)^{1/2}}{\sigma} \tag{4.4.5}$$

If we compare the maximum luminosity obtainable with the conductive flux to the total solar luminosity, we have

$$\frac{L_{\text{cond}}}{L_\odot} = \frac{4\pi R_\odot^2 F_{\text{cond}}}{L_\odot} \approx 10^{-12} \tag{4.4.6}$$

With the gradient estimated as $T_c/R_\odot$, using the central temperature to make the conductivity as large as possible and taking the geometric cross section to be about 10^{-20} cm^2, we still fail to carry the solar luminosity by at least 5 orders of magnitude. Thus, conduction can play no significant role in the energy transport in the sun. Indeed, that is true for all normal stars. However, in white dwarf stars, where the electrons are degenerate, the mean free path of the electrons is comparable to the dimensions of the star itself. Then conduction becomes so important that the internal temperature distribution is essentially isothermal.

If we combine equations (4.2.14) and (4.2.1) we can write the radiative flux as

$$F_{\mathrm{rad}} = \left(\frac{4ac}{3\bar{\kappa}\rho}\right) \nabla T(T^3) \tag{4.4.7}$$

which has the same form as equation (4.4.4). Thus we may define a conductive opacity from the conductivity so that

$$\kappa_{\mathrm{cond}} = \frac{4acT^3}{3\mathrm{K}\rho} \tag{4.4.8}$$

Then, if necessary, the conductive and radiative fluxes can be combined by augmenting the mean radiative opacity, so that

$$\frac{1}{\bar{\kappa}} = \frac{1}{\bar{\kappa}_{\mathrm{rad}}} + \frac{1}{\kappa_{\mathrm{cond}}} \tag{4.4.9}$$

4.5 Convective Stability

a Efficiency of Transport Mechanisms

We calculated the fluxes that can be transported by radiation, convection, and conduction, and we found that they produce rather different temperature gradients. However, we have seen from the integral theorems that the central temperature is set largely by the mass of the star, and in Chapter 3 we learned that the energy produced by nuclear processes will be a strong function of that temperature. Thus, virtually all the energy will be produced near the center and, in steady state, must make its way to the surface. In general, it will do this in the most efficient manner possible. That is, the mode of energy transport will be that which produces the smallest temperature gradient and also the greatest luminosity. In short, the star will choose among the methods available to it and select that which allows it to leak away its energy as fast as possible.

To carry enough energy to support the luminosity of the sun, conductive transport would require an immense temperature gradient. This is another way of saying that conduction is not important in the transport of energy. Convection will produce a temperature gradient which is nearly the adiabatic gradient and is fully capable of carrying all the energy necessary to sustain the solar luminosity. If we compare the radiative temperature gradient given in equation (4.2.14), and the adiabatic gradient as given in equation (4.3.19), we get

$$\frac{(\nabla T)_{\mathrm{rad}}}{(\nabla T)_{\mathrm{ad}}} = \frac{15\bar{\kappa}(r)\rho(r)L(r)k}{32\pi ac\mu m_h GM(r)T^3(r)} \sim 1 \tag{4.5.1}$$

From such an estimate the dominance of one mechanism over another is not obvious. Could both methods compete roughly equally? Or is it more likely that one method will prevail in part of the star, while the remainder will be the domain of the other. We have continually suggested that the latter is the case, and now we shall see the reason for this assertion.

b Schwarzschild Stability Criterion

For convection to play any role whatever, convective elements must be formed, and the conditions must be such that the elements will rise and fall. The statistical distribution law says that particles exist with the full range of velocities, and it would be remarkable if the particles were so uniformly distributed that any given volume had *exactly* the same number of particles of each velocity. This would be a very special particle distribution and not at all a random one. A random distribution would require that on some scale some volumes have more high-speed particles than others and hence can be considered to be hotter. In fact, an entire spectrum of such volumes will exist and can be viewed as perturbations to the mean temperature. Thus, the first of our conditions for convective transport will always be met. Temperature fluctuations will always exist. But will they result in elements that move? In developing an expression for the adiabatic gradient, we assumed that the convective element will expand adiabatically and so do no work on the surrounding medium. This is certainly the most efficient way the element can move, and it cannot be exactly met in practice. To move, the element must displace the material ahead of it. There must be some "viscous" drag on the element requiring the element to do "work" on the surrounding medium. So the adiabatic expansion of a convective element is clearly the "best it can do" in getting from one place to another. Let us see if we can quantify this argument.

Let us assume that the gas is an ideal gas, and for the reasons mentioned above we assume that the element will behave adiabatically. Under these conditions we know that, the element will follow a polytropic equation of state, namely,

$$P \propto \rho^{\gamma} \qquad \gamma = \tfrac{5}{3} \tag{4.5.2}$$

Now consider a volume element which is displaced upward and has state variables denoted with an asterisk, while the surrounding values are simply P, T, and ρ (see Figure 4.2).

If $\rho_2^* \geq \rho_2$ then the element will sink or will not have risen in the first place. Initially, we require the conditions at point 1 to be the same (we are displacing the element in an ad hoc manner). Thus,

$$P_1^* = P_1 \qquad T_1^* = T_1 \qquad \rho_1^* = \rho_1 \tag{4.5.3}$$

Adiabatic expansion of the element requires that pressure equilibrium be maintained throughout the displacement, so

$$P_2^* = P_2 \tag{4.5.4}$$

We may express the conditions at point 2 in terms of a Taylor series and the conditions at point 1 so that

$$P_2^* = P_2 = P_1 + \frac{dP}{dr}dr + \cdots +$$

$$\rho_2 = \rho_1 + \frac{d\rho}{dr}dr + \cdots + \tag{4.5.5}$$

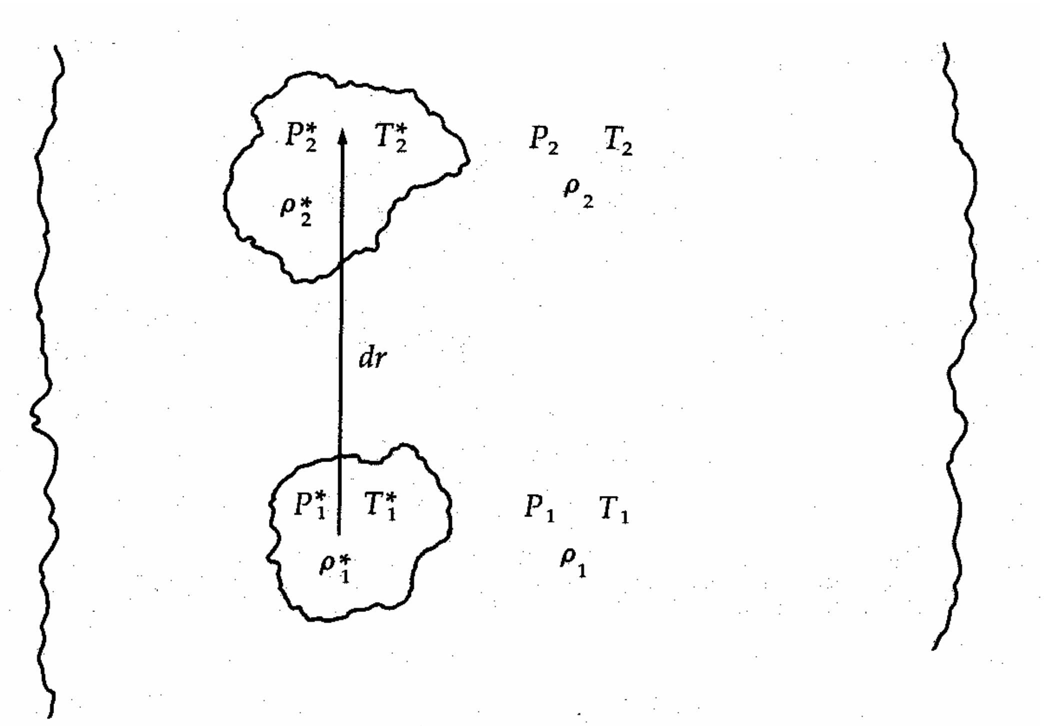

Figure 4.2 shows a schematic representation of a convective element with state variables denoted by * and surrounded by an ambient medium characterized by state variables P,T, and ρ. The element is initially at position 1 and is displaced through a distance dr to position 2.

Using the equation of state, we may write

$$\rho_2^* = (\text{const})(P_2^*)^{1/\gamma} = (\text{const})(P_2)^{1/\gamma} = \rho_1\left(1 + \frac{1}{P_1}\frac{dP}{dr}dr + \cdots +\right)^{1/\gamma}$$
$$\approx \rho_1\left(1 + \frac{1}{\gamma P_1}\frac{dP}{dr}dr + \cdots +\right) \tag{4.5.6}$$

If we take $\rho_2^* \geq \rho_2$ to be a condition for stability (i.e., the element will return to its initial position if displaced), then equations (4.5.5), and (4.5.6) require that

$$\frac{1}{\gamma P}\frac{dP}{dr} \geq \frac{1}{\rho}\frac{d\rho}{dr} \tag{4.5.7}$$

The ideal-gas law requires that

$$\frac{1}{P}\frac{dP}{dr} = \frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{T}\frac{dT}{dr} \tag{4.5.8}$$

which can be used to replace the density gradient in inequality (4.5.7) to get

$$\left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} \leq \frac{dT}{dr} \tag{4.5.9}$$

Dividing by dT/dr, we obtain the Schwarzschild stability criterion for a polytropic gas

$$\frac{d\ln P}{d\ln T} \geq \frac{\gamma}{\gamma - 1} = n + 1 \tag{4.5.10}$$

which for a monatomic gas with a $\gamma = 5/3$ is just

$$\frac{d\ln P}{d\ln T} \geq 2.5 \tag{4.5.11}$$

Thus, if the logarithmic derivative of pressure with respect to temperature is greater than or equal to 2.5, convection will not occur. In other words, if the actual temperature gradient is less than the adiabatic gradient, convection will not occur. This, then, is our means for deciding whether convection or radiation will be the dominant mode of energy transport. Should radiation be able to transport the energy with a temperature gradient less than the adiabatic gradient, no energy will be carried by convection, for the gas is stable against the thermal perturbations which must exist. However, if this is not the case, convection will be established; and it is so very efficient that it is capable of carrying all the energy with a temperature gradient that is just slightly super-adiabatic. For most of stellar structure, we may regard energy transport as being bimodal; either radiation or convection will transport the energy, with the decision being made by equation (4.5.10). The Schwarzschild stability criterion has been shown to be quite general and will hold under the most varied of conditions, including those stars where general relativity must be included to describe their structure.

4.6 Equations of Stellar Structure

Having settled the mode of energy transport, we are in a position to describe the structure of a star in a steady-state condition. This is a good time to review briefly what we have done. The equations of stellar structure arise from conservation laws and relationships developed from the local microphysics. In Chapter 1, we posed the basic problem of stellar interiors to be the description of the variation of state variables P, T, and γ with position in the star. For spherical stars, this amounts to indicating their dependence on the radial coordinate r. In developing that description, we introduced additional variables and their relation to the state variables so that by now our list of parameters has grown to nine members, P(r), T(r), p(r), M(r), L(r), ε(r), $\overline{\kappa}$ (r), γ(r),and μ(r). To specify these parameters, we have at our disposal three conservation laws and a transport equation in addition to three functional relationships derived from the microphysics. The function γ(r) can also be specified by microphysics and is usually given by its adiabatic value. Only the variation of μ(r) needs to be specified *ab initio*. When we move to the stage of evolving the stellar models, the chemical composition will need to be specified for the initial model since the processes of nuclear energy generation will tell us how the composition changes with time. However, we must, at least initially, specify both the composition of the star and how it varies throughout the entire star. The use of a convective theory of transport which attempts to improve on the adiabatic gradient will also introduce another parameter, known as the mixing length, which must also be specified *ab initio*.

The constraints posed by the conservation laws take the form of differential equations whose solution is subject to a set of boundary conditions. Below is a summary of these differential equations and their origin:

$$\begin{array}{ll}
(a)\ \dfrac{dM(r)}{dr} = 4\pi r^2 \rho(r) & \text{conservation of mass, eq. (2.1.8)} \\[2ex]
(b)\ \dfrac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r) & \text{conservation of energy, eq. (4.2.16)} \\[2ex]
(c)\ \dfrac{dP(r)}{dr} = -\dfrac{GM(r)\rho(r)}{r^2} & \text{conservation of momentum, eq. (2.1.6) (hydrostatic equilibrium)} \\[2ex]
\quad\ \dfrac{dT(r)}{dr} = -\dfrac{3\bar{\kappa}(r)\rho(r)L(r)}{16\pi ac T^3(r) r^2} & \text{radiative transport, eq. (4.2.4)} \\[2ex]
(d)\ \left(\dfrac{dT}{dr}\right)_{ad} = -\dfrac{\mu m_h GM(r)}{(n+1)kr^2} & \text{convective transport, eq. (4.3.19)} \\[2ex]
\quad\ (\Delta\nabla T) = \left[\dfrac{L^2(r)T(r)}{C_p^2\rho^2(r)GM(r)\pi^2 l^4 r^2}\right]^{1/3} & \text{eq. (4.3.18)}
\end{array} \tag{4.6.1}$$

In addition to these differential equations we have the following relations from the microphysics:

$$\begin{array}{ll}
\epsilon = \epsilon[T(r), \rho(r), \mu(r)] & \text{nuclear energy production (Chap. 3)} \\
\bar{\kappa} = \bar{\kappa}[T(r), \rho(r), \mu(r)] & \text{radiative opacity (Sec. 4.1)} \\
\gamma = \gamma[T(r), \rho(r), \mu(r)] = \frac{5}{3} & \text{(Chap. 2)} \\
P = P[T(r), \rho(r), \mu(r)] & \text{equation of state (Chap. 1)}
\end{array} \tag{4.6.2}$$

These eight relationships and the chemical composition completely specify the structure of the star. We now turn to describing methods by which their solution can be obtained.

4.7 Construction of a Model Stellar Interior

The construction of stellar models in steady state is essentially a numerical procedure which has been the subject of study of a large number of astrophysicists since the early 1950s and the pioneering work of Harm and Schwarzschild[2]. Basically two methods have been employed to solve the equations. The early work utilized a scheme described by Schwarzschild which amounts to a straightforward numerical integration of the differential equations of stellar structure. In the early 1960s, this procedure was superceded by a method due to Henyey which replaces the differential equations with a set of finite difference equations whose solution is carried out globally and enables one to include time-dependent phenomena in a natural way. However, since this method requires an initial solution which is usually obtained by the Schwarzschild procedure, we describe both methods.

a Boundary Conditions

Using the functional relations given by equations (4.6.2), we may reduce the problem of solving the structure equations to one of finding solutions for the four differential equations given in equations (4.6.1). These constitute a set of four nonlinear first-order differential equations in four unknowns. In general, such a system will have four constants of integration which must be specified to guarantee a solution. In principle, two of these constants are specified by requiring that the model be physically reasonable. These are

$$\begin{aligned} M(r) &\to 0 \qquad \text{as } r \to 0 \\ L(r) &\to 0 \qquad \text{as } r \to 0 \end{aligned} \tag{4.7.1}$$

At the other end of the range of the independent variable,

$$\begin{aligned} M(r) &\to M_* \qquad \text{as } r \to R_* \\ L(r) &\to L_* \qquad \text{as } r \to R_* \end{aligned} \tag{4.7.2}$$

However, five constants are specified by equations (4.7.1), and (4.7.2), if R_* is included as a parameter. Only four of these can be linearly independent. Thus, if one specifies M_* and R_*, the solution will specify L_*. Another aspect of the problem is that the constants are not all specified at the same boundary, and so it is not possible to treat the problem as an initial-value problem and to solve by straightforward numerical integration. Such problems are known as *two-point boundary-value problems*, and one must essentially guess the missing integration constants at one boundary, obtain the numerical solution complete to the other boundary, and adjust the guesses until the specified integration constants at the far boundary are obtained. A further problem arises from the fact that the equations of hydrostatic equilibrium and energy transport are numerically unstable as $r \to 0$ because the derivatives require the calculation of "0/0" at the origin. However, the problem can be recast as a double-eigenvalue problem with the fitting (solution adjustment) taking place in the interior but away from the boundary. This is essentially the Schwarzschild approach.

b Schwarzschild Variables and Method

When one is searching for the numerical solution to a physical problem, it is convenient to re-express the problem in terms of a set of dimensionless variables whose range is known and conveniently limited. This is exactly what the Schwarzschild variables accomplish. Define the following set of dimensionless variables:

$$\begin{aligned} x &= \frac{r}{R_*} & & \\ q &= \frac{M(r)}{M_*} & p &= P\left(\frac{GM_*^2}{4\pi R_*^4}\right)^{-1} \\ f &= \frac{L(r)}{L_*} & t &= T\left(\frac{m_h \mu G M_*}{k R_*}\right)^{-1} \end{aligned} \tag{4.7.3}$$

Note that the first three variables are the fractional radius, mass, and luminosity, respectively, while the two at the right represent the pressure and temperature normalized by a constant which describes the way they vary homologously. In addition, let us assume that the opacity and energy generation rate can be approximated by

$$\bar{\kappa} = \kappa_0 \rho^n T^{-s} \qquad \epsilon = \epsilon_0 \rho^\lambda T^\nu \tag{4.7.4}$$

The differential equations of stellar structure then become

$$\begin{array}{lll} (a) & \dfrac{dp}{dx} = -\dfrac{qp}{tx^2} & \text{hydrostatic equilibrium} \\[2ex] (b) & \dfrac{dq}{dx} = \dfrac{px^2}{t} & \text{mass conservation} \\[2ex] (c) & \dfrac{df}{dx} = D(\lambda, \nu) x^2 p^{\lambda+1} t^{\nu-\lambda-1} & \text{energy conservation} \\[2ex] & \dfrac{dt}{dx} = \dfrac{-C(n, s) f p^{n+1}}{x^2 t^{n+s+4}} & \text{radiative equilibrium} \\[1ex] (d) & & \\[1ex] & \dfrac{d\ln p}{d\ln t} = 2.5 & \text{convective equilibrium} \end{array} \tag{4.7.5}$$

which are subject to the boundary conditions

$$q(0) = f(0) = 0 \qquad q(1) = f(1) = 1 \tag{4.7.6}$$

The parameters C(*n*,s) and D(λ,ν) are the eigenvalues of the problem, and these values specify the type of star being considered. In physical variables they are

$$C(n, s) = \frac{3}{4(4\pi)^{n+2}ac}\left(\frac{k}{m_hG}\right)^{s+4}\frac{\kappa_0}{\mu^{s+4}}\frac{L_*R_*^{s-3n}}{M_*^{s-n+3}}$$

$$D(\lambda, \nu) = \frac{1}{(4\pi)^{\lambda}}\left(\frac{m_hG}{k}\right)^{\nu}(\epsilon_0\mu^{\nu})\frac{M_*^{\nu+\lambda+1}}{L_*R_*^{\nu+3\lambda}} \tag{4.7.7}$$

Note that the ideal-gas law has been used to eliminate the density from the problem, and this may cause some problems with the solution at the surface where the pressure and temperature essentially go to zero, in addition to the numerical problems at the center when x → 0. However, Schwarzschild shows that near the surface one may approximate the dimensionless pressure *p* and dimensionless temperature *t* by

$$t \approx \frac{n+1}{n+7.5}\,\frac{1-x}{x}$$

$$p \approx (\text{const})t^{(n+7.5)/(n+1)} \qquad \text{for radiative envelopes} \tag{4.7.8}$$

If the star has a convective core, then all the energy is produced in a region where the structure is essentially specified by the adiabatic gradient and so the energy conservation equation [equation (4.7.5c)] is redundant. This means that the eigenvalue D(λ,ν) is unspecified and the problem will be solved by determining C(*n*,s) alone. Such a model is known as a *Cowling model*. The additional constraints on the solution are specified by the mass and size of the convective core (q_c and x_c). These are determined by the value of x for which d(ln *p*)/d(ln t) < 2.5, and the star becomes subject to the radiative temperature gradient. The stellar luminosity is then $L = L_c$ and for the envelope f = 1. While such a scheme works well for models with convective cores, numerical problems will generally occur at the center should it be in radiative equilibrium and the solution obtained numerically. However, a slightly different set of dimensionless variables can be defined where the pressure and temperature are scaled by their values at the center of the star. The differential equations of stellar structure become stable at r = 0 since the dimensionless pressure and temperature are both unity at the center by definition. One then, integrates outward from the center with P_c and T_c as eigenvalues. The stellar mass, luminosity, and radius can be related to these new eigenvalues. That there are two distinct eigenvalues is demonstrated by the surface boundary condition that both the surface pressure and surface temperature must vanish at the same value of x.

Unfortunately the equations of stellar structure become numerically unstable near the surface for the same reasons that required the approximation of the solutions of equations (4.7.5) by equation (4.7.8). Although the errors in the model can be made small with the aid of modern computers, it is bad practice to numerically solve equations which are inherently unstable. For that reason, the usual procedure is to integrate from both the outside and the inside and to make the fit at the boundary between the core and envelope. The approximations near the surface are still present, but their effect on the solution is minimized. In actual practice, the fitting can be accomplished in the U-V plane where the solutions are homologously invariant. The fitting procedure is similar to that described in Chapter 2.

Since Schwarzschild introduced this method of solution of the equations of stellar structure in the 1950s, many variants have been used by numerous investigators. In one form or another, all variants suffer from problems similar to those that plague the Schwarzschild procedure. In general, this approach to the numerical solution of two-point nonlinear boundary-value problems always suffers from the propagation of errors from one boundary to the other. The most serious of these errors are usually the truncation errors associated with the numerical integration scheme which tend to be systematic. However, this approach enabled the generation of stellar models which represented the steady-state aspect of stars for the first time. Although qualitative information about stellar evolution can be gained from polytropes (and we do so in Chapter 5), specific and detailed descriptions of stellar evolution require the generation of steady-state models. However, some aspects of stellar evolution happen on time scales which are very short compared to the thermal time scale, and in some instances short compared to the dynamical time scale. Often, substantive changes occur to the internal structure which produces only small changes at the surface. Thus, minor changes in the surface boundary conditions can reflect monumental changes in the internal structure of the star. In addition, we must include the time-dependent terms in the equations describing the conservation of momentum and energy. Specifically, if some of the generated energy does work on the star, causing it to expand as energy is liberated by contraction, then this energy must be included in the energy conservation equation relating the stellar luminosity to the sources of energy. This is usually accomplished by keeping track of the time rate of change of the entropy. The direct integration scheme does not readily lend itself to the inclusion of such terms. Such models are no longer merely steady-state models, and we will require more sophisticated tools to deal with them.

c Henyey Relaxation Method for Construction of Stellar Models

To overcome some of the numerical instability problems described in the previous section, Louis Henyey et al[3]. developed a superior numerical scheme in the early 1960s. This method is the foundation for all modern stellar-model calculations. His approach was to transform the problem to a set of variables in which the nonlinearity of the differential equations was minimized. The differential equations of stellar structure were then replaced with a set of finite difference equations whose solution could be carried out simultaneously over the entire model. This tended to reduce the effect of truncation error by spreading it more or less evenly across the model. Furthermore, the addition of time-dependent terms proved to be relatively easy to incorporate in the structure equations. We do not describe all the details that make this method so powerful, but only sketch the principles involved.

We begin by replacing the independent variable r with M(r). Henyey noticed that the behavior of the equations was far more linear when the mass interior to r was used as the independent variable. The radial coordinate then becomes a dependent variable whose value must be found for any particular M(r). If we make this transformation, the four differential equations of stellar structure become

$$\begin{aligned}
&\frac{\partial P}{\partial M(r)} = \frac{GM(r)}{4\pi r^4} && \textbf{hydrostatic equilibrium}\\
&\frac{\partial r}{\partial M(r)} = (4\pi r^2 \rho)^{-1} && \textbf{conservation of mass}\\
&\frac{\partial L}{\partial M(r)} = \epsilon(P, T, \mu) - T\frac{\partial S}{\partial t} && \textbf{conservation of energy}\\
&\frac{\partial \ln T}{\partial \ln P} = f(P, T, \mu) && \textbf{energy transport}
\end{aligned} \qquad (4.7.9)$$

Here we have explicitly included the time dependent entropy term in the energy equation for purposes of example. In addition, we have written the energy transport term in a general manner which can accommodate either radiation or convection. Now we divide the star into N - 1 zones, starting with the center as the first point and ending at the surface or some outer point where the boundary conditions are known. By approximating the derivatives of equations (4.7.9) by the difference of the parameters at adjacent points, we get the following finite difference equations:

$$\begin{aligned}
\frac{P_{i+1}-P_i}{M_{i+1}-M_i} &= \frac{GM_{i+1/2}}{4\pi r^4_{i+1/2}} \\
\frac{r_{i+1}-r_i}{M_{i+1}-M_i} &= (4\pi r^2_{i+1/2})^{-1} P^{-1}_{i+1/2} \\
\frac{L_{i+1}-L_i}{M_{i+1}-M_i} &= \varepsilon_{i+1/2} - T_{i+1/2}\left.\frac{\partial S}{\partial t}\right|_{i+1/2} \\
\frac{T_{i+1}-T_i}{P_{i+1}-P_i} &= \frac{T_{i+1/2}}{P_{i+1/2}} f(P_{i+1/2}, T_{i+1/2}, M_{i+1/2})
\end{aligned} \tag{4.7.10}$$

The subscript i + ½ is used exclusively on the right-hand side of equations (4.7.10) to indicate that the value to be used is intermediate between the values at i and i + 1. It will turn out that we must have an initial guess of the model's structure in order to solve the finite difference equations. It is this guess which may supply the initial information for evaluating the parameters at the points i + ½. Since the mass points M_i represent the independent variable of this problem, the four equations given in equations (4.7.10) contain eight unknowns. However, we have N - 1 systems of such equations with considerable overlap in unknowns among them. The situation at the outer zone will be handled somewhat differently since there is no N + 1 point. Thus if we count the total number of equations we have 4N - 4. But at each point there are only four unknowns, making the total number of unknowns of 4N. The remaining four constraints are essentially the boundary conditions of the problem. By analogy to the Schwarzschild problem, let us take the central boundary conditions to be $r_1 = L_1 = 0$, which removes two of the additional unknowns. Now if we choose two of the remaining unknowns at the surface, such as r_N and L_N, the problem is completely specified. Indeed, if we choose the surface pressure to be zero, then choosing a star of a particular mass and radius (and distribution of chemical composition) will specify the stellar configuration. One of the motivating notions that led Henyey to this type of technique was the ability to match a stellar interior to a model of the stellar atmosphere. This technique is ideally suited to do this. One simply takes as the outer zone that point where the physical parameters are known as the result of a separate study. In the second part of this book, we present a theory of stellar atmospheres which provides far more accurate surface boundary conditions than those of early investigators. In addition to improving the manner in which the surface boundary conditions are handled, it may be advisable to ignore the point at the center. A Taylor series expansion can be used to express the values of P_2, T_2, L_2, and r_2 in terms of the central temperature T_c and P_c. Because the system of equations is strongly diagonal, the solution is easier to come by if the central boundary conditions are expressed in this manner.

The Henyey approach shown in equations (4.7.10) represents the derivative of the structure equations by first-order finite differences. Thus the errors of the approximation are second order in those derivatives. This necessitates the use of the large number of zones to accurately represent the model, and it is this large number of zones that represents the primary computational burden in the construction of the model. Although increasing the order of the finite difference equations would improve the stability, it would also increase the density (i.e., the number of terms) of the resulting linear algebraic equations, slowing their solution and decreasing their stability. Budge[4] has shown that an improvement in the accuracy of the approximation can be achieved by using a Runge-Kutta fourth-order approximation for the derivatives without increasing the resulting linear equation density. Although there is some increase in the computational burden for obtaining the coefficients, this is more than offset by being able to reduce the number of zones in the model.

We still must solve these linear equations. It is not uncommon in the standard Henyey scheme to choose up to 500 zones in the star, which will yield some 2000 nonlinear equations in as many unknowns. Now it is clear why we need an initial solution. If we have a solution which is close to the correct one, we may express the correct solution in terms of the initial solution and a small linear correction to that solution. This will reduce the system of nonlinear equations to a linear system where the corrections are the unknowns. Such a scheme is known as a *Newton-Raphson iteration scheme*. Since the system is sparse (each equation contains only 8 of the 2000 unknowns) and the independent variable was chosen so as to make the equations somewhat linear, the iteration scheme is usually stable. However, the stability also depends on the quality of the initial solution. This is normally obtained by means of a Schwarzschild-type integration or a previously determined model.

It is clear that the Henyey method lends itself naturally to the problem of stellar evolution. In this case the initial model is a model calculated for an earlier time. Thus the procedure would be to start at some initial time with a Schwarzschild model, allow a small interval of time for the model to pass by, calculate the changes in the chemical composition resulting from nuclear processes, and modify the model accordingly. This serves as the initial first guess for the Henyey scheme, and a new model is produced. The effects of time are again allowed for, and the next Henyey model is constructed, etc. In this way an entire sequence of stellar models representing the life history of the star can be constructed. One generally starts the sequence when the star is well represented by a steady-state model, and the Schwarzschild solution gives an accurate description of the stellar structure. Such a time is the arrival on the main sequence and the accompanying onset of hydrogen burning. The resulting life history of the star is as good as the microphysics which goes into the solution and the quality of the computer and the associated numerical techniques used to obtain the solution.

At this point, we have covered the fundamentals required to construct a model of the interior of a star. However, we should not leave the impression that such a model would reflect the accuracy of contemporary stellar interior models. There are many complications and refinements which should be treated and included to produce a model with modern sophistication. We have said nothing about the small departures of the equation of state from the ideal-gas law which occur at quite modest densities due to electron screening. Nor have we dealt with many of the vagaries of the theory of convection, such as semi-convection, convective overshoot, or mixing-length determination. These result largely from the primitive nature of the existing theory of convection, and while they do pose significant problems at certain points in a star's evolution, they do not affect the conceptual picture of stellar structure. It seems almost criminal not to devote more attention to the efforts of those who have labored to provide improved opacities and nuclear energy generation rates. But again, while these improve the details of the models and enhance our confidence in the predictions based on them, they do not conceptually change the basic physics upon which the models rest. While we have outlined the numerical procedures necessary to actually solve the structure equations, there is much cleverness and imaginative numerical analysis required to translate what we outlined to a computer program which will execute to completion in an acceptable time. Do not forget that the early models of Schwarzschild and Harm were calculated basically by hand, aided only by a desk calculator whose capabilities are far exceeded by even the cheapest pocket calculator of the present. It is no accident that the rapid advance of our knowledge of stellar structure parallels the explosive advance in our ability to carry out numerical calculations.

An understanding of the refinements of contemporary models is essential for any who would choose to do meaningful research in stellar interiors. It is not essential for those who would understand the results and their physical motivation, and it is to those people that this book is addressed. With the knowledge of the physical processes that determine the structure of stars, let us now turn to the crowning achievement of the study of stellar structure - the theory of stellar evolution.

Problems

1. Assume that there is a star in which the energy is generated uniformly throughout the star (that is, ε = constant). In addition, the opacity is constant (i.e., electron scattering). Further assume that the star is in radiative and hydrostatic equilibrium. Show that $1-\beta$ is constant throughout the star and that it is a polytrope of index $n = 3$.

2. Suppose that in a star, the only source of energy generation is radioactive decay, so the energy production per unit mass is constant and independent of density and temperature. Further suppose that the opacity is given by Kramer's' law. Show that the structure of the star is described by a polytrope, and find the value of the polytropic index *n*.

3. Compare the local value of the radiative gradient with the adiabatic gradient throughout the sun. Describe the regions of radiative and convective equilibrium in light of your results. What would you expect to be the effect on the radiative and convective zones of replacing 50 percent of the solar luminosity with an energy generation source which was more efficient?

4. For a sphere in radiative equilibrium and STE, show that the radiation pressure is one third the energy density.

5. Since the convective temperature gradient differs systematically from the adiabatic gradient, it is possible that the cumulative effect is significant when it is integrated over the entire convective zone. Examine this effect in the sun and decide whether it is significant.

6. Use a stellar interiors code or existing models to find the variation of the fractional ionization of hydrogen and helium with depth in the sun.

7. Use a stellar interiors code, or existing models to find the fraction by mass and radius within which (a) 20 percent, (b) 50 percent, (c) 90 percent, and (d) 99 percent of the sun's energy is generated.

8. Repeat Problem 7 for a star of $10M_\odot$.

9. Repeat problem 7 but with $\mathbf{Z} = 10^{-8}$.

10. Determine the relative importance of bound-bound transitions, bound-free transitions, and electron scattering as opacity sources in the sun.

References and Supplemental Reading

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