# 5

# *Theory of Stellar Evolution*

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One of the great triumphs of the twentieth century has been the detailed description of the life history of a star. We now understand with some confidence more than 90 percent of that life history. Problems still exist for the very early phases and the terminal phases of a star's life. These phases are very short, and the problems arise as much from the lack of observational data as from the difficulties encountered in the theoretical description. Nevertheless, continual progress is being made, and it would not be surprising if even these remaining problems are solved by the end of the century.

To avoid vagaries and descriptions which may later prove inaccurate, we concentrate on what is known with some certainty. Thus, we assume that stars can contract out of the interstellar medium, and generally we avoid most of the detailed description of the final, fatal collapse of massive stars. In addition, the fascinating field of the evolution of close binary stars, where the evolution of one member of the system influences the evolution of the other through mass exchange, will be left for another time. The evolution of so-called normal stars is our central concern.

Although the details of the theory of stellar evolution are complex, it is possible to gain some insight into the results expected of these calculations from some simple considerations. We have developed all the formalisms for calculating steady-state stellar models. However, those models could often be accurately represented by an equilibrium model composed of a polytrope or combinations of polytropes. We should then expect that the evolutionary history of a star could be approximately represented by a series of polytropic models. What is needed is to find the physical processes relating one of these models to another thereby generating a sequence. Such a description is no replacement for model calculation for without the details, important aspects of stellar evolution such as lifetimes remain hidden. In addition, there are branching points in the life history of a star where the path taken depends on results of model calculations so specific that no general considerations will be able to anticipate them. However, a surprising amount of stellar evolution can be understood in terms of sequences of equilibrium models connected by some rather general notions concerning the efficiency of energy transfer. Descriptions of these models, and their relationship to one another, form the outline upon which we can hang the details of the model calculations.

In general, we trace the evolution of a star in terms of a model of that star's changing position on the Hertzsprung-Russell diagram. With that in mind, let us briefly review the range of parameters which define the internal structure of a star.

## 5.1 The Ranges of Stellar Masses, Radii, and Luminosity

In Section 2.2 b, we used the $\beta^*$ theorem to show that as the mass of a star increases, the ratio of radiation pressure to total pressure also increases so that by the time one reaches about $100M_\odot$ approximately 80 percent of the pressure will be supplied by the photons themselves. Although it is not obvious, at about this mass the outer layers can no longer remain in stable equilibrium, and the star will begin to shed its mass. Very few stars with masses above $100M_\odot$ are known to exist, and those that do show instabilities in their outer layers. At the other end of the mass scale, a mass of about $0.1M_\odot$ is required to produce core temperatures and densities sufficient to provide a significant amount of energy from nuclear processes. Thus, we can take the range of stellar masses to span roughly 3 powers of 10 with the sun somewhat below the geometric mean.

If we include the white dwarfs, which may be the size of the earth or less, the range of observed stellar radii is about 6 powers of 10 with the sun again near the geometric mean. The observed range of stellar surface temperatures is by far the smallest being from about 2000 K for the coolest M star, to perhaps 50000 K for some O-type stars. If the temperature range is combined with the range in radii, it is clear that we could expect a range as great as 17 powers of 10 in the luminosity. In practice, the largest stars do not have the highest temperatures, so that the range in luminosity is nearer 10 powers of 10. A reasonable range of these parameters is then

$$\begin{aligned} 10^{-1}M_\odot &\le M_* \le 10^2 M_\odot \\ 10^{-3}R_\odot &\le R_* \le 10^3 R_\odot \\ 10^{-4}L_\odot &\le L_* \le 10^6 L_\odot \end{aligned} \tag{5.1.1}$$

This, then, represents the ranges of the defining parameters of those objects we call normal stars. The theory of stellar evolution will tell us which parameters are related to various aspects of a star's life.

## 5.2 Evolution onto the Main Sequence

### a Problems concerning the Formation of Stars

Since we began this discussion with the assumption that stars form by contraction from the interstellar medium, honesty requires that we describe several forces that mitigate against that contraction. For a star to form by gravitational contraction from the interstellar medium, all sources of energy which support the initial cloud must be dominated by gravity. For the typical interstellar cloud with sufficient mass to become a star, we shall see that not only is this not true of the collective sources of energy, but also it is not true of them individually.

***The Internal Thermal Energy*** From the Virial theorem as derived in Chapter 1 [equation (1.2.34)], the internal kinetic energy of the gas of the cloud must be less than one-half the gravitational energy in order for the moment of inertia to show any accelerative contraction. Thus for a uniform density gas at a certain temperature T, the mass must be confined inside a sphere of a certain radius $R_c$. That radius can be found from

$$2\left(\frac{3\rho kT}{2\mu m_h}\right)\left(\frac{4\pi R_c^3}{3}\right) \le \frac{GM^2}{R_c} \tag{5.2.1}$$

or

$$R_c \le \frac{GM\mu m_h}{3kT} \approx \frac{0.25(M/M_\odot)}{T} \quad \text{pc} \tag{5.2.2}$$

This distance is sometimes known as the Jeans length, for it is the distance below which a gas cloud becomes gravitationally unstable to small fluctuations in density. For a solar mass of material with a typical interstellar temperature of 50 K, the cloud would have to be smaller than about $5\times10^{-3}$ pc with a mean density of about $10^8$ particles per cubic centimeter. This is many orders of magnitude greater than that found in the typical interstellar cloud, so it would seem unlikely that such stars should form.

***The Rotational Energy*** The Virial theorem can also be used to determine the effects of rotation on a collapsing cloud. Again, from Chapter 1, the rotational kinetic energy must be less than one-half the gravitational potential energy in order for the cloud to collapse. So

$$2(\tfrac{1}{2}I\omega^2) \le \frac{GM^2}{R_c} \tag{5.2.3}$$

which for a sphere of uniform density and constant angular velocity gives

$$R_c \le \left(\frac{5GM}{2\omega^2}\right)^{1/3} \tag{5.2.4}$$

The differential rotation of the galaxy implies that there must be a shear or velocity gradient which would impart a certain amount of rotation to any dynamical entity forming from the interstellar medium. For an Oort constant, A = 16 km/s/kpc, this implies that

$$R_c \le 0.9\left(\frac{M}{M_\odot}\right)^{1/3} \quad \text{pc} \tag{5.2.5}$$

Thus, it would seem that to quell rotation, the initial mass of the sun must have been confined within a sphere of about 0.7 pc.

***Magnetic Energy*** A similar argument concerning the magnetic energy density M, where

$$M = \frac{H^2}{8\pi}\frac{4\pi R_c^3}{3} \tag{5.2.6}$$

can be made by appealing to the Virial theorem with the result that

$$R_c \le 0.37\left(\frac{M}{M_\odot H_{\mu g}}\right)^{1/2} \quad \text{pc} \tag{5.2.7}$$

For a value of the ambient interstellar galactic magnetic field of 5 microgauss we get

$$R_c \leq 0.17\left(\frac{M}{M_\odot}\right)^{1/2} \quad \text{pc}$$
(5.2.8)

How are we to reconcile these impediments to gravitational contraction with the fact that stars exits? One can use the rotational and magnetic energies against one another. A moderate magnetic field of a spinning object will cause a great deal of angular momentum per unit mass to be lost by a star through the centripetal acceleration of a stellar wind. The resulting spin-down of the star will weaken the internal sources of the stellar magnetic field itself. Observations of extremely slow rotation among the magnetic $A_p$ seem to suggest that this mechanism actually occurs. Clouds can be cooled by the formation of dust grains and molecules, as long as the material is shielded from the light of stars by other parts of the cloud. The high densities and low temperatures observed for some molecular clouds imply that this cooling, too, is occurring in the interstellar medium. However, unless some sort of phase transition occurs in the material, the thermal cooling time is so long that it is unlikely that the cloud will remain undisturbed for a sufficient time for the Jeans' condition to be reached. Thus it seems unlikely that the Jeans' condition can be met for low-mass clouds.

It is clear from equation (5.2.2) that $R_c \sim \sqrt{T/\rho}$, so for a given temperature the Jeans' length increases with decreasing density. However, the Jean's mass increases as the cube of the Jeans' length. Thus, for a cloud of typical interstellar density to collapse, it must be of the order of $10^4$ $M_\odot$. It is thought that the contraction of these large clouds creates the conditions enabling smaller condensations within them to form protostars. The pressure that the large contracting cloud exerts on smaller internal perturbations of greater density may squeeze them down to within the Jeans' length after which these internal condensations unstably contract to form the protostars of moderate mass. These are some of the arguments used to establish the conditions for gravitational contraction upon which all stellar formation depends, and since stars do form, something of this sort must happen.

### b Contraction out of the Interstellar Medium

Since we have given some justification for the assumption that stars will form out of clouds of interstellar matter which have become unstable to gravitational collapse, let us consider the future of such a cloud.

***Homologous Collapse*** For simplicity, consider the cloud to be spherical and of uniform density. The equation of motion for a unit mass of material somewhere within the cloud is

$$\frac{d^2r}{dt^2} = -\frac{GM(r)}{r^2}$$
(5.2.9)

If we assume that the material at the center doesn't move [that is, $\mathbf{v}(0) = 0$], then the first integral of the equations of motion yields

$$v^2 = \frac{8\pi G\rho_0}{3}\int_0^r r\,dr = \frac{4\pi G\rho_0}{3}r^2 \qquad (5.2.10)$$

or

$$v \propto r \qquad (5.2.11)$$

This says that at any time the velocity of *collapse* is proportional to the radial coordinate. This is a self-similar velocity law like the Hubble law for the expansion of the universe, only in reverse. Thus, at any instant the cloud will look similar to the cloud at any other point in time, only smaller and with a higher density $\rho_0$. Thus, the density will remain constant throughout the cloud but steadily increase with time. Since the velocity is proportional to r, the collapse is homologous and we obtain Lane's law of Chapter 2 [equation (2.3.9)] which completely specifies the internal structure throughout the collapse.

One should not be left with the impression that this homologous collapse is uniform in time. It is not; rather, it proceeds in an accelerative fashion, resulting in a rapid compaction of the cloud. When the density increases to the point that internal collisions between particles produce a pressure sufficient to oppose gravity, the equations of motion become more complicated. Some of the energy produced by the collapse leaks away in the form of radiation from the surface of the cloud and a temperature gradient is established. These processes destroy the self-similar, or homologous, nature of the collapse, and so we must include them in the equations of motion.

However simple and appealing this solution may be, it is a bit of a swindle. The mathematics is correct, and the assumption that v(0) = 0 may be quite reasonable. However, it is unlikely that most clouds are spherically symmetric and of uniform density. Density fluctuations must exist and without the pressures of hydrostatic equilibrium to oppose the central force of gravity, there is no *a priori* reason to assume spherical symmetry. Normally this would seem like unnecessary quibbling with an otherwise elegant solution. Unfortunately, these perturbations are amplified by the collapse itself and destroy any possibility of the cloud maintaining a uniform density.

***Non-Homologous Collapse*** Let us consider the same equations of motion as before so that the first integral is given by

$$\int_{v_i^2}^{v^2} dv^2 = 2\int_{r_i}^{r}\frac{4\pi}{3}G\rho_0 r\,dr = 2\int_{r_i}^{r}\frac{GM(r)\,dr}{r^2} \qquad (5.2.12)$$

Now we wish to follow the history of a point within which the mass is constant so that M(r) is constant and

$$v^2(R) - v_i^2 = 2GM(R)\left(\frac{1}{R} - \frac{1}{R_i}\right) \tag{5.2.13}$$

The variable R has replaced r, and this introduces a minus sign into equation (5.2.13), since for a collapse dr = -dR. Equation (5.2.13) is just the energy integral, so there are no surprises here. Now we change variables so that

$$x = \frac{R}{R_i} \qquad \alpha = \frac{2GM(R_i)}{R_i^3} \qquad u = \frac{v}{R_i} \tag{5.2.14}$$

If we take the initial velocity of the cloud to be zero and the initial value of x to be 1, then equation (5.2.13) becomes

$$u^2 = \left(\frac{dx}{dt}\right)^2 = \frac{\alpha(1-x)}{x} \qquad 0 \le x \le 1 \tag{5.2.15}$$

which can be integrated over time to give

$$t(x) = \alpha^{-1/2}\left[\frac{\pi}{2} - \mathrm{Sin}^{-1}\sqrt{x} + (x - x^2)^{1/2}\right] \tag{5.2.16}$$

Now a can be related to the mean density, so that we can rewrite equation (5.2.16) as

$$t\left(\frac{R}{R_i}\right) = \left[\frac{3}{8\pi G\overline{\rho_i(R_i)}}\right]^{1/2}\left\{\frac{\pi}{2} - \mathrm{Sin}^{-1}\left(\frac{R}{R_i}\right)^{1/2} + \left[\frac{R}{R_i} + \left(\frac{R}{R_i}\right)^2\right]^{1/2}\right\} \tag{5.2.17}$$

If $\langle\rho_i(R_i)\rangle$ is constant and not a function of $R_i$, then we recover the homologous contraction which is clearly not uniform in time. However, if the initial mean density is a decreasing function of $R_i$, then the collapse time of a sphere of material $M(R_i)$ is an increasing function of $R_i$. This means that initial concentrations of material will become more concentrated and any inhomogeneities in the density will grow unstably with time.

This is essentially the result found by Larson[1] in 1969. If the cloud is gravitationally confined within a sphere of the Jeans' length, the cloud will experience rapid core collapse until it becomes optically thick. If the outer regions contain dust, they will absorb the radiation produced by the core contraction and reradiate it in the infrared part of the spectrum. After the initial free-fall collapse of a $1M_\odot$ cloud, the inner core will be about 5 AU surrounded by an outer envelope about 20000 AU When the core temperature reaches about 2000 K, the $H_2$ molecules dissociate, thereby absorbing a significant amount of the internal energy. The loss of this energy initiates a second core collapse of about 10 percent of the mass with the remainder following as a "heavy rain". After a time, sufficient matter has rained out of the cloud, and the cloud becomes relatively transparent to radiation and falls freely to the surface, producing a fully convective star. While this scenario seems relatively secure for low mass stars (i.e., around $1M_\odot$), difficulties are encountered with the more massive stars. Opacities in the range of 1500 to 3000 K make the evolutionary tracks somewhat uncertain. Indeed, there are some indications that massive stars follow a more homologous and orderly contraction to the state where they become fully convective.

Although this is the prevailing picture for the early phases of the evolution for low-mass protostars, there are some difficulties with it. Such stars would be shielded from observation by the in-falling rain of material until quite late in their formation. Since the entire configuration including the rain is hardly in a state of hydrostatic equilibrium, the arguments given below would not pertain until quite late in the star's formation, by which time the star may well have reached the main sequence. There seems to be little support in observation for this point of view, and the entire subject is still somewhat controversial.

Michael Disney[2] has pointed out that the details of the collapse from the interstellar medium depend critically on the ratio of the sound travel time to the free-fall time in the contracting protostar. Although this ratio is typically unity [equations (3.2.4), (3.2.6), and (3.2.9)], small departures from unity appear to matter. The free-fall time is basically the time during which the collapse takes place, and the sound travel time is the time required for the interior to sense the effects of pressure disturbances initiated at the boundary. Thus, if $\tau_s/\tau_f > 1$, the interior tends to be unaffected by the boundary pressure during the collapse. Any external pressure will then tend to compress the matter in the outer part of the collapsing cloud without affecting the interior regions, removing any density gradients that may exist in the perturbation and forcing the collapse to be more nearly homologous. This would reduce the effect of the rain and cause the protostar to collapse more as a unit. Any initial velocity resulting from of the homologous collapse of the large cloud will only exacerbate the situation by significantly shortening the time required for the collapse. Thus the initial phases of star formation remain in some doubt and probably depend critically on the circumstances surrounding the initial conditions of the collapse of the larger cloud.

### c Contraction onto the Main Sequence

Once the protostar has become opaque to radiation, the energy liberated by the gravitational collapse of the cloud cannot escape to interstellar space. The collapse will slow down dramatically and the future contraction will be limited by the star's ability to transport and radiate the energy away into space. Initially, it was thought that such stars would be in radiative equilibrium and that the future of the star would be dictated by the process of radiative diffusion in the central regions of the star. Indeed, for most stars this is true for the phases just prior to nuclear ignition. However, Hayashi[3] showed that there would be a period after the central regions became opaque to radiation during which the star would be in convective equilibrium.

***Hayashi Evolutionary Tracks*** In Chapter 4 we found that once convection is established, it is incredibly efficient at transporting energy. Thus, as long as there are no sources of energy other than gravitation, the future contraction will be limited by the star's ability to radiate energy into space rather than by its ability to transport energy to the surface. We have also learned that the structure of a fully convective star will essentially be that of a polytrope of index $n = 1.5$. We may combine these two properties of the star to approximately trace the path it must take on the Hertzsprung-Russell diagram.

With gravitation as the only source of energy and the contraction taking place on a time scale much longer than the dynamical time, the Virial theorem allows one-half of the change in gravitational energy to appear as the luminosity and be radiated away into space. The other half will go into the internal energy of the star increasing the internal temperature. Thus,

$$L = \frac{1}{2}\frac{d(GM^2/R)}{dt} = -\frac{\frac{1}{2}GM^2}{R^2}\frac{dR}{dt} \tag{5.2.18}$$

Since the luminosity is positive, dR/dt must be negative which ensures that the star will contract. Since the luminosity is related to the surface parameters by

$$L = 4\pi R_*^2 \sigma T_e^4 \tag{5.2.19}$$

the change in the luminosity with respect to the radius will be

$$\frac{dL}{dR_*} = \frac{4L}{T_e}\frac{dT_e}{dR_*} + \frac{2L}{R_*} \tag{5.2.20}$$

Equation (5.2.19) is essentially a definition of what we mean by the *effective temperature*. As long as the star remains in convective equilibrium, it will be a polytrope and the contraction will be a self-similar, and thus homologous, contraction.

Since the rate of stellar collapse is dictated by the photosphere's ability to radiate energy, we should expect the photospheric conditions to dictate the details of the collapse. Indeed, as we shall see in the last half of this book, the eigenvalues that determine the structure of a stellar atmosphere are the surface gravity and the effective temperature. So as long as the stellar luminosity is determined solely by the change in gravity, and the energy loss is dictated by the atmosphere, we might expect that the independent variable $T_e$ to remain unchanged. However, it is necessary to show that such a sequence of models actually forms an evolutionary sequence. The extent to which this will be true depends on the radiative efficiency of the photosphere. This is largely determined by the opacity. At low temperatures the opacity will increase rapidly with temperature owing to the ionization of hydrogen. This implies that any homological increase of the polytropic boundary temperature at the base of the atmosphere will be met by an increase in the radiative opacity and a steepening of the resultant radiative gradient. This increase in the radiative opacity also forces the radiating surface farther away from the inner boundary, causing the effective temperature to remain unchanged. A much more sophisticated argument demonstrating this is given by Cox and Giuli[4].

The star can effectively be viewed as a polytrope wrapped in a radiative blanket, with the changing size of the polytrope being dictated by the leakage through the blanket. The blanket is endowed with a positive feedback mechanism through its radiative opacity, so that the effective temperature remains essentially constant. The validity of this argument rests on the ability of convection to deliver the energy generated by the gravitational contraction efficiently to the photosphere to be radiated away. With this assumption, we should expect the effective temperature to remain very nearly constant as the star contracts. Thus $dT_e/dR_*$ in equation (5.2.20) will be approximately zero, and we expect the star to move vertically down the Hertzsprung-Russell (H-R) diagram with the luminosity changing roughly as $R_*^2$ until the internal conditions within the star change. Thus for the Hayashi tracks

$$\frac{dT_e}{dR_*} = \frac{dT_e}{dL} = 0 \qquad \frac{d\ln L}{d\ln R_*} = +2 \tag{5.2.21}$$

While the location of a specific track will depend on the atomic physics of the photosphere, the relative location of these tracks for stars of differing mass will be determined by the fact that the underlying star is a polytrope of index $n = 3/2$. From the polytropic mass-radius relation developed in Chapter 2 [equation (2.4.21)] we see that

$$M^{1/3}R = \text{constant} \tag{5.2.22}$$

and that

$$\frac{d\ln R_*}{d\ln M} = -\frac{1}{3} \tag{5.2.23}$$

Equations (5.2.19), and (5.2.20) also imply that

$$\frac{dL}{dM} = \frac{2L}{R_*}\frac{dR_*}{dM} + \frac{4L}{T_e}\frac{dT_e}{dM} \tag{5.2.24}$$

If we inquire as to the spacing of the vertical Hayashi tracks in the H-R diagram, then we can look for the effective temperatures for stars of different mass but at the same luminosity. Thus, we can take the left-hand side of equation (5.2.24) to be zero and combine the right-hand side with equation (5.2.23) to get

$$\frac{d\ln T_e}{d\ln M} = +\frac{1}{6} \tag{5.2.25}$$

This extremely weak dependence of the effective temperature on mass means that we should expect all the Hayashi tracks for the majority of main sequence stars to be bunched on the right side of the H-R diagram. Since the star is assumed to be radiating as a blackbody of a given $T_e$ and is in convective equilibrium, no other stellar configuration could lose its energy more efficiently. Thus no stars should lie to the right of the Hayashi track of the appropriate mass on the H-R diagram; this is known as the *Hayashi zone of avoidance*.

We may use arguments like these to describe the path of the star on the Hertzsprung-Russell diagram followed by a gravitationally contracting fully convective star (see Figure 5.1). As we suggested, this contraction will continue until conditions in the interior change as a result of continued contraction.

As the star moves down the Hayashi track, the internal temperature will increase in a homologous fashion so that T = μM/R. Hence we could expect the adiabatic gradient $\nabla T_{ad} = \mu M/R^2$. However, the radiative gradient is

$$\frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{16\pi acT^3r^2} \sim \frac{\kappa(M/R^3)L}{(\mu M/R)^3R^2} \sim \frac{\kappa L}{\mu^3M^2R^2} \tag{5.2.26}$$

To find the homological behavior of the radiative opacity, we may use the approximate formulas [equation (4.1.19)] for Kramer's-like opacity. Making use of the homology transformations for p and T we can calculate the ratio of the adiabatic to radiative gradient $\mathfrak{R}$ as

$$\mathfrak{R} \sim \frac{M^{s-n+3}}{L(R^{s-3n})} \tag{5.2.27}$$

As the star contracts down the Hayashi track, $\mathfrak{R}$ will steadily increase. At some point, depending on the dominant source of opacity, the adiabatic gradient will exceed the radiative gradient, and convection will cease. This will not happen globally all at once; rather, a radiative core will form that propagates outward until the entire star is radiative. At that point the mode of collapse will change because the primary barrier to energy loss will move from the photosphere to the interior and the diffusion of radiant energy. Since all the models on the Hayashi tracks are convective polytropes, we might expect this point to happen at the same value of $\mathfrak{R}$ for stars of differing mass. If this is the case, then, remembering that for stars on the Hayashi tracks L . $R^2$, we may use equation (5.2.27) to find that the locus of points of constant $\mathfrak{R}$ lies along a line such that

$$\frac{d\ln L}{d\ln M} = 2\left(\frac{s-n+3}{s-3n+2}\right) \tag{5.2.28}$$

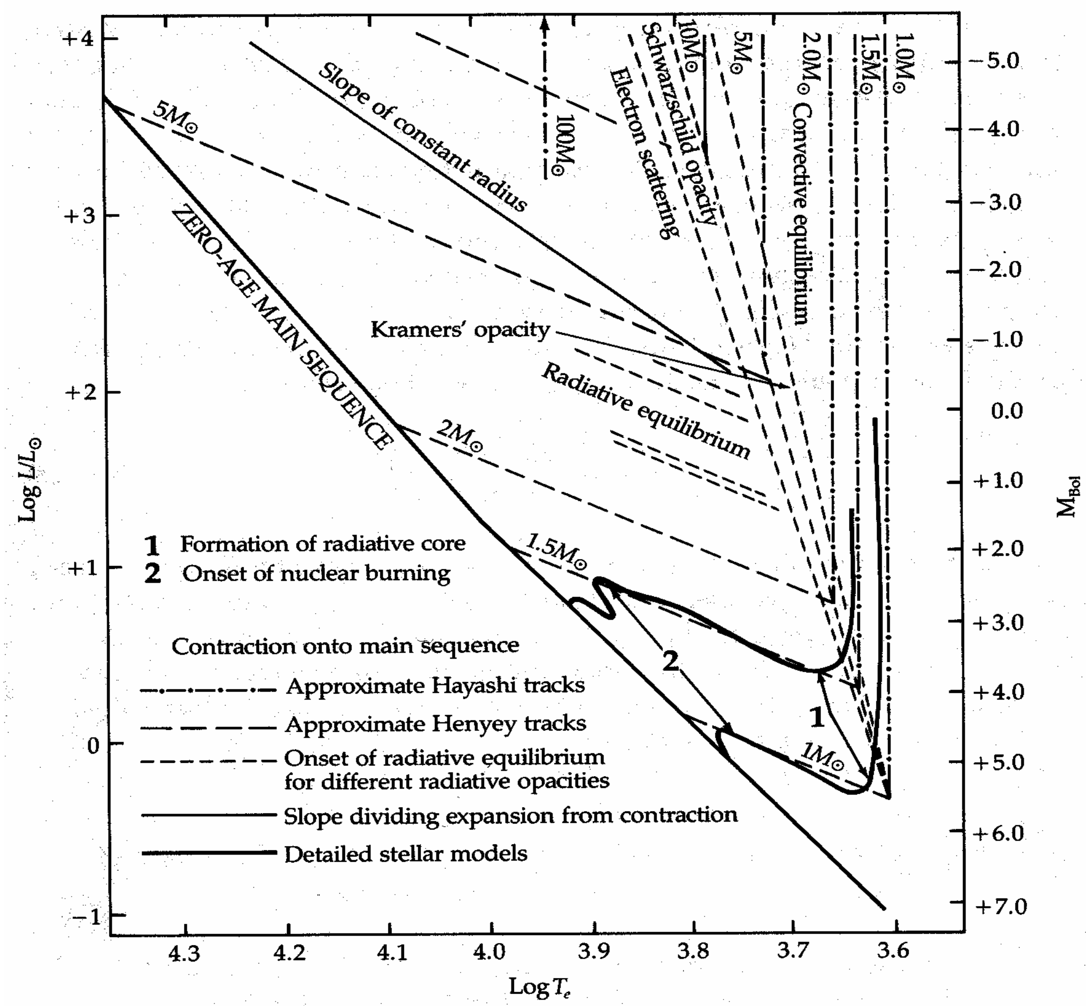


**Figure 5.1** shows the schematic tracks for fully convective stars and radiative stars on their way to the main sequence. The low dependence of the convective tracks on mass implies that most contracting stars will occupy a rather narrow band on the right hand side of the H-R diagram. The line of constant radius clearly indicates that stars on the Henyey tracks continue to contract. The dashed lines indicate the transition from convective to radiative equilibrium for differing opacity laws. The solid curves represent the computed evolutionary tracks for two stars of differing mass[5].

Remembering that for electron scattering $n = s = 0$, while for Kramers' opacity $s = 7/2$ and $n$ is 1 or 0.75, depending on the relative dominance of free-free to bound-free opacity, we can obtain the appropriate mass luminosity law for the dominant source of opacity at the point of transition from convective to radiative equilibrium. Combining this with equation (5.2.25), we find that the locus of points in the H-R diagram will be described by

$$\frac{d\ln L}{d\ln T_e} = \begin{cases} +18 & \text{electron scattering} \\ +21.24 & \text{Schwarzschild opacity for bound-bound opacity} \\ +26.4 & \text{Kramers' opacity for free-free opacity} \end{cases} \tag{5.2.29}$$

For the very massive stars, radiation pressure may play an important role toward the end of the Hayashi contraction phase, so that the onset of radiative equilibrium occurs sooner, increasing the value on the right-hand side of equation (5.2.28) slightly. But for stars with a mass less than about $3M_\odot$ equations (5.2.25), and (5.2.29) will describe their relative position on the H-R diagram with some accuracy.

***Henyey Evolutionary Tracks*** After sufficient time has passed for the adiabatic gradient to exceed the radiative gradient, convection ceases and the main barrier to energy loss is no longer the ability of the photosphere to radiate energy into space. Rather the radiative opacity of the core slows the leakage of energy generated by gravitational contraction, and the atmosphere no longer provides the primary barrier to the loss of energy. Further contraction now proceeds on the Kelvin-Helmholtz time scale. As the star continues to shine, the gravitational energy continues to become more negative, and to balance it, in accord with the Virial theorem, the internal energy continues to rise. This results in a slow but steady increase in the temperature gradient which results in a steady increase in the luminosity as the radiative flux increases. This increased luminosity combined with the ever-declining radius produces a sharply rising surface temperature as the photosphere attempts to accommodate the increased luminosity. This will yield tracks on the H-R diagram which move sharply to the left while rising slightly (see Figure 5.1). For the reasons mentioned above, the beginning of these tracks will be along a series of points which move upward and to the left for stars of greater mass.

We may quantify this by asking how the luminosity changes in time. We differentiate equation (5.2.18) and obtain

$$\frac{dL}{dt} = -\frac{1}{2}\frac{d^2\Omega}{dt^2} = -\frac{aGM^2}{2R^2}\left[-\frac{2}{R}\left(\frac{dR}{dt}\right)^2 + \frac{d^2R}{dt^2}\right] \tag{5.2.30}$$

The parameter a is simply a measure of the central condensation of the model, which we require to be independent of time. This requirement is satisfied if the contraction is homologous. If we further invoke the Virial theorem and require that the contraction proceed so as to keep the second derivative of the moment of inertia equal to zero, then

$$\frac{d^2I}{dt^2} = \frac{d^2}{dt^2}\left(\alpha MR^2\right) = 0 \Rightarrow \left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt^2} = 0 \tag{5.2.31}$$

Using this to replace the second derivative in equation (5.2.30), we get

$$\frac{d\ln L}{d\ln R} = -3 \tag{5.2.32}$$

Multiplying equation (5.2.24) by (R/L)(dM/dR) and combining with the above result, we get

$$\frac{d\ln T_e}{d\ln R} = -\frac{5}{4} \qquad \frac{d\ln L}{d\ln T_e} = +\frac{12}{5} \tag{5.2.33}$$

Thus we can expect the star to move upward and to the left on the H-R diagram with a slope of 2.4. This path will eventually carry it to the main sequence where nuclear burning will set in as a consequence of the steady increase in the central temperature. Since the effect of onset of nuclear burning will be similar for a wide range of main sequence stars, we can expect the luminosity distribution of the Henyey tracks with mass to be reflected on the main sequence. As a result, we might expect that equation (5.2.28) will reflect the main sequence Mass-Luminosity relation. Indeed, Harris, Strand, and Worley[6] give empirical values of 2.76 for the exponent on the mass of the mass-luminosity relation for the lower main sequence and 4 for the upper main sequence. The values obtained from equation (5.2.28) corresponding to different types of opacity are

$$\frac{d\ln L}{d\ln M} = \begin{cases} +3 & \text{electron scattering} \\ +3.54 & \text{Schwarzschild's opacity} \\ +4.4 & \text{Kramers' opacity} \end{cases} \tag{5.2.34}$$

Although the proper evolutionary tracks for a star contracting to the main sequence require more exact modeling than can be done with polytropes, the overall effects can be estimated by considering sequences of equilibrium configurations linked by a description of those physical processes which limit the energy flow from the star. To dramatize this, we included in Figure 5.1 some calculated evolutionary tracks for two stars. The salient features of the pre-main sequence evolution are reasonably described by the curves in spite of the crude assumptions involved.

## 5.3 The Structure and Evolution of Main Sequence Stars

When the track of a gravitationally contracting star intersects the main sequence, the star has reached a point in its life when it will be stable for an extended period of time. This is ensured observationally by the fact that about 90 percent of all stars reside on or very near the main sequence and so must be involved in the utilization of their most prolific and efficient source of energy – the fusion of hydrogen into helium. So we may take the intersection of the Henyey track with the main sequence as an indication that hydrogen ignition has begun in the stellar core.

Actually nuclear processes begin somewhat before the main sequence is encountered. The first constituents of the star to undergo nuclear fusion are deuterium and lithium which require conditions substantially below that of hydrogen for their ignition. However, their abundance is sufficiently low so that they provide little more than a stabilizing effect on the star as it proceeds along its Henyey track, causing the star to hook on to the main sequence.

For the sun, about a million years is required for the equilibrium abundances of the proton-proton cycle to be established with sufficient accuracy for their use in the energy generation schemes. At this point, the star can be said to have arrived at the zero-Age main sequence. As the name implies, this is generally taken as the beginning point of stellar evolution calculations, as the onset of nuclear burning makes the details of the prior evolution largely irrelevant to the subsequent evolution. In some real sense, the star forgets where it came from. Since it is fairly obvious that the effects of stellar evolution during the main sequence phase will result in little movement on the H-R diagram, we need to understand more of the structure of the interior to appreciate these effects. Therefore, we begin by describing the structure to be expected for the hydrogen burning models that describe the main sequence. The structure of main sequence stars can be readily broken into two distinct groups: those that occupy the upper half of the main sequence, and those that occupy the lower main sequence.

### a Lower Main Sequence Stars

We define the *lower main sequence* to be those stars with masses less than about 2 solar masses. For these stars, after the trace elements with low ignition temperatures have been exhausted and hydrogen fusion has begun, the equilibrium structure is established in about a million years. The mass of these stars is insufficient to produce a central temperature high enough to initiate the CNO cycle, so the primary source of energy is the proton-proton cycle. Models indicate that in the sun, 98 percent of the energy is supplied by the proton-proton cycle. The relatively low dependence on temperature of the proton-proton cycle implies that the energy generation will be less concentrated toward the center than would be the case with the CNO cycle. This and the modest central temperature imply that a temperature gradient less than the adiabatic gradient is all that is required to carry the energy produced by the p-p cycle. Thus, these stars have a central core which is in radiative equilibrium. However, in the sun, the adiabatic gradient is never far from the actual temperature gradient, and it would take a very little increase in the temperature gradient to cause the core to become unstable to convection. Indeed, the conditions for convective instability are met in the outer regions of these stars resulting in the formation of a convective envelope. In the sun, this point is reached at about $0.75R_\odot$, so that about 98.8 percent of the mass is included in the radiative core. Ultimately, the situation is reversed near the surface, as it must be, for the energy leaves the surface of the star by radiation.

The existence of the radiative core in stars of the lower main sequence has a significant effect on the subsequent evolution of the star. The $^4$He, which is the end product of hydrogen burning, remains in the locale in which it is produced. However, since the production rate is strongly dependent on temperature, the helium abundance increases more rapidly as one approaches the center of the star. The helium must be supported against its own gravity while it contributes nothing to the support of the remainder of the star. As a result, the internal temperature will increase to maintain the luminosity in the face of decreasing hydrogen abundance and the increasing mass of the particles (i.e. the $^4$He). This is why the temperature scales with the mean molecular weight m, [see equation (2.3.8)]. Thus, we should expect stars like the sun to slowly increase in brightness, as the internal temperature rises, during their main sequence lifetime. Indeed, the standard solar model indicates that the solar luminosity has increased by about 40 percent since its arrival on the zero age main sequence.

Toward the end of the star's main sequence life, the helium abundance will rise to the point where a core of helium, surrounded by a hydrogen burning shell, will form in the center of the star. The support of this isothermal helium core is eventually helped by the Pauli Exclusion Principle. In Chapter 1, we outlined the equation of state to be expected for a gas where all the available $h^3$ volumes of phase space were filled. Because of their lower mass, this condition will be experienced first by the electrons. The degenerate equation of state does not contain the temperature and therefore permits the existence of an electron pressure capable of assisting in the support of the helium core; this equation is independent of the conditions existing in the hydrogen-burning shell. Thus as the core builds, we could expect its structure to shift from that of an isothermal sphere, described in Chapter 2, to that of a polytrope with a $\gamma = 5/3$, as would be dictated by the electron pressure of a fully degenerate gas. This change from an isothermal sphere to a polytrope will dictate the mass distribution, for the pressure of the ions becomes small compared to that of the electrons. However, because of the high conductivity of a degenerate gas, the configuration will remain isothermal since any energy surplus can immediately be transported to a region of energy deficit by electron conduction. Thus, the region is still known as the isothermal helium core, even though the pressure equilibrium is dictated by the electron pressure behaving as a polytropic gas with a $\gamma$ of 5/3.

Therefore, the main sequence lifetime of a low mass star consists of a steady energy output from hydrogen burning in an environment of steadily increasing helium. On a nuclear time scale, the helium abundance increases preferentially in the most central regions causing the temperature to rise which results in a slow increase in the luminosity throughout the main sequence lifetime of the star. After about 10 percent of the radiative core mass has been consumed, an isothermal helium core begins to form and structural changes begin to occur very rapidly. This signals the end of the main sequence lifetime.

### b Upper Main Sequence Stars

The situation regarding the stellar structure for stars of more than 2 solar masses is nearly reversed from that of the lower main sequence. For stars on the main sequence, the observed mass-radius relation is approximately

$$M \propto R^{4/3} \tag{5.3.1}$$

However, from the homology relations in Chapter 2 [equations (2.3.8)], we know that

$$T \propto M/R \tag{5.3.2}$$

Therefore, for stars along the main sequence, we expect the central temperature to increase slowly as we proceed up the main sequence in accord with

$$T_c \propto R^{1/3} \propto M^{1/4} \tag{5.3.3}$$

This slow rise in the central temperature will result in a greater fraction of the energy being produced by the more temperature-sensitive CNO cycle. Thus, by the time one reaches stars of greater than about 2 solar masses, the CNO cycle will be the dominant source of energy production. The much larger temperature sensitivity of the CNO cycle as compared to the p-p cycle means that the region of energy production will be rather more centrally concentrated than in stars of less mass. This, in turn, requires a steeper temperature gradient in order to transport the energy to the outer parts of the star. Since in the sun the radiative gradient was already quite close to the adiabatic gradient, this small increase is sufficient to cause the inner regions to become convectively unstable, and a substantial convective core will be established. However, in the outer parts of the star, the declining density causes the product of $\bar{\kappa}\rho$, which appears in the radiative gradient [equation (4.2.14)], to reduce the radiative gradient below that of the adiabatic gradient, and so convection stops. Thus, we have a star composed of a convective core surrounded by an envelope in radiative equilibrium. This role reversal for the core and envelope has a profound effect on the evolution of the star.

The presence of a convective core ensures that the inner regions of the star will be well mixed. As helium is produced from the burning of hydrogen, it is mixed thoroughly throughout the entire core. Thus, we do not have a buildup of a helium core that increases in helium abundance toward the center in these stars. Instead, the entire convective core is available as a fuel source for energy production at the center of the star. For this reason, energy production is remarkably steady in these stars until the entire convective core is nearly exhausted of hydrogen. Even as exhaustion approaches, the extreme temperature dependence of the CNO cycle implies that deficits produced by the declining availability of hydrogen fuel can be made up by modest increases in the temperature and hence minor changes in the structure of the star. Indeed, it is not until more than 99 percent of the convective core mass has been converted to helium that truly significant changes occur in the structure of the star and the star can be said to be leaving the main sequence.

## 5.4 Post Main Sequence Evolution

The evolution of stars off the main sequence represents the response of the star to a depletion of the available fuel supply, and it can be qualitatively understood by examining the response of the core and envelope to the attempts of the nuclear burning regions to adjust to the diminution of the available hydrogen. In stars of the lower main sequence, the hydrogen-burning shell begins to move into a region of declining density resulting in a decrease in available hydrogen. For stars of the upper main sequence, the situation is somewhat different. The convective nature of the core ensures the existence of mass motions, which continue to bring hydrogen into the central regions for hydrogen burning until the entire core is depleted. These two rather different approaches to hydrogen exhaustion produce somewhat different evolutionary futures for the two kinds of stars, so we examine them separately.

### a Evolution off the Lower Main Sequence

The development of a helium core, which signals the onset of post main sequence evolution, is surrounded by a thin hydrogen-burning shell. The hydrogen burning continues in a shell around the helium core which steadily grows outward, in mass, through the star. However, the helium core must be supported against its own gravity as well as support the weight of the remaining star, and its energy sources are all on the outside. As a result, it is impossible for the hydrogen-burning shell to establish a temperature gradient within the helium core. Only gravitational contraction of the core will result in the release of energy inside the helium core, and except for this source of energy the helium core must be isothermal, with its temperature set by the burning of hydrogen surrounding it. But the rate of hydrogen burning is dictated largely by the mass of material lying above the burning zone, because this is the material that must kept in equilibrium. As the mass of the isothermal helium core increases, the equilibrium temperature of the core will also rise and this demand can be met only by a slow contraction of the helium core. The slow increase in the core temperature triggers a steady increase in the stellar luminosity.

As the isothermal core grows through the addition of He from the hydrogen-burning shell, the core temperature must rise in order for it to remain in equilibrium and support the outer layers of the star. Since an isothermal sphere is a unique polytropic configuration, it seems reasonable that there would be a limit to the amount of overlying material that such an isothermal core could support. This limit is known as the *Chandrasekhar-Schönberg (C-S) limit*. The limit will depend solely on the mass fraction of the isothermal core and the mean molecular weights of the core and envelope. Should the core exceed this limiting mass fraction, it must contract to provide the temperature and pressure gradients necessary to support the remainder of the star as well as itself.

***Chandrasekhar-Schönberg Limit*** A detailed evaluation of the Chandrasekhar-Schönberg limit requires matching the isothermal core solution to the pressure required to support the overlying stellar mass. The maximum mass fraction that an isothermal core can have is

$$q_{\text{C-S}} \approx 0.37\left(\frac{\mu_o}{\mu_i}\right)^2 \tag{5.4.1}$$

where $\mu_o$ and $\mu_i$ are the mean molecular weight of the outer region and core respectively. Although the specific calculation of $q_{\text{c-s}}$ requires detailed consideration of the isothermal core solution, we can provide an argument for the plausibility of such a limit by considering the Virial theorem for the core alone.

$$3(\gamma - 1)U_c + \Omega_c - 3P\left(\frac{4\pi r_c^3}{3}\right) = 0 \tag{5.4.2}$$

where

$$U_c = \frac{kTM_c}{\mu_i m_h}$$
$$\Omega_c \approx -\frac{GM_c^2}{r_c} \tag{5.4.2a}$$

and $r_c$ is the radius of the isothermal core. The third term on the left-hand side arises because we cannot take the volume integrals, which yield the global theorem, over a surface where the pressure is zero. Thus, we must include a "surface" term which is effectively the surface pressure times the enclosed volume of the core. We may solve this expression for the pressure at the boundary of the core and obtain

$$P(r_c) = \frac{1}{4\pi r_c^3}\left[\frac{3(\gamma - 1)kTM_c}{\mu_i m_h} - \frac{GM_c^2}{r_c}\right] \tag{5.4.3}$$

Now we wish to find the maximum core radius which will provide sufficient pressure to support the remaining star. We can find a maximum pressure by differentiating equation (5.4.3) with respect to $r_c$ and finding that value of $r_c$ for which the pressure gradient is zero. Certainly any core which yields a zero surface pressure gradient is the largest physically reasonable core. This calculation results in a maximum $r_c$ given by

$$\frac{1}{r_c} = \frac{9(\gamma - 1)kT}{4\mu_i m_h G M_c} \tag{5.4.4}$$

Substitution into equation (5.4.3) yields the maximum surface pressure attainable at the surface of the core.

$$P_m(r_c) = 3\left[\frac{9(\gamma - 1)}{4}\right]^4 \left(\frac{kT}{\mu_i m_h}\right)^4 G^{-3} M_c^{-2} \tag{5.4.5}$$

Remember that the homologous behavior of the temperature allows us to write

$$T = (\text{const})\left(\frac{\mu_o M}{R}\right) \tag{5.4.6}$$

so we can eliminate the temperature from 5.4.5 and express the result with a term which is homologous to the pressure of the envelope. Thus,

$$P_m(r_c) = (\text{const})\left(\frac{T^4}{\mu_i^4 M_c^2}\right) = (\text{const})\left(\frac{M^2}{R^4}\right)\left(\frac{\mu_o}{\mu_i}\right)^4 \left(\frac{M}{M_c}\right)^2 \tag{5.4.7}$$

The term $M^2/R^4$ is homologous to the pressure of the envelope at the surface of the core. The ratio of this to the maximum attainable core pressure must be less than unity for the core to be able to support the envelope,

$$P_m(r_c) = (\text{const})\left(\frac{T^4}{\mu_i^4 M_c^2}\right) = (\text{const})\left(\frac{M^2}{R^4}\right)\left(\frac{\mu_o}{\mu_i}\right)^4 \left(\frac{M}{M_c}\right)^2 \tag{5.4.8}$$

where the constant is the same as in equation (5.4.1) . Thus, we see that the isothermal core can, at most, support about 37 percent of the mass of the star, but if the core is primarily helium, this limit is reduced to about 10 percent.

***Degenerate Core*** Only for stars near the upper end of our range (i.e. $M \approx M_\odot$) will the mass of the core approach the Chandrasekhar-Schönberg limit without becoming degenerate and the core undergoing further gravitational contraction. For stars with masses $M \leq 1.3M_\odot$ the slowly developing isothermal core will be degenerate from a point in its development when the core mass is well below the Chandrasekhar-Schönberg limit. Under these conditions, that limit does not apply because the added pressure of the degenerate electron gas is sufficient to support nearly any additional mass. Thus, the isothermal helium cores of lower main sequence stars can increase to virtually any mass below the Chandrasekhar degeneracy limit. As mass is added to the core, we can expect the core to contract according to the mass-radius law for degenerate configurations that we derived in Chapter 1 [equation (1.3.18)]. This law, following differentiation with respect to time, indicates that the core will shrink on the same time scale that mass is added to it, and that is the nuclear time scale.

***Progress to the Red Giant Phase*** In terms of its physical size, this isothermal degenerate helium core is never very large. Thus the post main sequence evolution of a low-mass star can be viewed as the processing of stellar material through the burning zone, with the resultant helium being packed into a very small volume of systematically higher mean molecular weight. The declining density just above the helium core will lead to an increase in temperature, in order for the nuclear energy generation mechanisms to supply the energy required to support the star. However, an increase in the central temperature would lead to an increase in the temperature gradient and an increase in the luminosity. The increased luminosity, in turn, causes the outer envelope of the star to expand, decreasing the temperature gradient. Equilibrium is established at a higher shell temperature and somewhat greater luminosity and temperature gradient. The result is that the star moves upward and very slightly to the right on the H-R diagram. The process continues until the temperature gradient exceeds the adiabatic gradient. Then the entire outer envelope becomes convective. The increase in physical size of the envelope lowers the surface temperature and thereby increases the radiative opacity in the outer layers. This further decreases the efficiency of radiative transport and hastens the formation of the outer convection zone. The outer envelope is now well approximated by a polytrope of index $n = 1.5$, and the conditions for the Hayashi tracks become operative.

The star now approximately follows the track of a fully convective star only now in reverse. The continual decline of the available hydrogen supply in the shell burning region, which becomes extremely thin, leads to a steady increase in the shell temperature and accompanying rise in the luminosity. With the outer convection zone behaving as a good polytrope and efficiently carrying the energy to the surface, the energy loss is again limited by the photosphere and the star expands rapidly to accommodate the increased energy flow. The star now moves nearly vertically up the giant branch as a red giant.

***Helium Flash*** As the temperature of the hydrogen-burning shell increases and the degenerate core builds in mass; the temperature eventually reaches approximately $10^8$ K. This is about the ignition temperature of helium via the triple-$\alpha$ processes. Under normal conditions, the burning of helium could begin in a measured way which would allow for an orderly transition of nuclear energy generation processes. However, the core is degenerate, so the electron pressure is only weakly dependent on temperature. Indeed, the limiting equation of state for total degeneracy does not contain the temperature at all. Thus helium burning sets in with unrelenting ferocity. With its extreme dependence on temperature, the triple-$\alpha$ process initiates a thermal runaway which is limited only by the eventual removal of the degeneracy from the core. The complete equation of state for a partially degenerate gas does indeed, contain the temperature and at a sufficiently high temperature the equation of state will revert to the ideal-gas law. When this occurs, the core rapidly expands, cools, and reaches equilibrium, with helium continuing to burn to carbon in its center. The response of the core to this entire process is so swift that the total energy produced is a small fraction of the stored energy of the star. In addition, the site for the production of the energy is sufficiently far removed from the outer boundary that energy is diffused smoothly throughout the star and never makes a noticeable change in the stars appearance.

The duration of the flash, is so much shorter time than the dynamical time scale for the entire star that one would expect that all manifestation of the flash would be damped out by the overlying star and remain hidden from the observer. However, detailed hydrodynamical calculations[7] indicate that the pressure pulse resulting from the rapid expansion of the core arrives at the surface with a velocity well in excess of the escape velocity. This may well result in a one-time mass loss of the order of 30 percent which would affect the subsequent evolution.

***Terminal Phases of Low Mass Evolution*** Initially, after helium ignition, the hydrogen burning shell continues to supply about 90 percent of the required support energy. However, now an orderly transition of energy mechanisms can take place, resulting in the transfer from hydrogen burning to helium burning over an extended time. The star will move somewhat down the giant branch and out on the horizontal branch, from near the peak of the giant branch where the helium flash took place. Meanwhile the helium core is in convective equilibrium, with the convection zone extending almost to the hydrogen shell. The re-expansion of the core is responsible for the contraction of the outer envelope, causing the star to move out onto the horizontal branch. It appears likely, that after helium burning has ceased and the resultant carbon core is contracting, the outer envelope becomes unstable to radiation pressure and lifts off the star, forming a planetary nebula and leaving the hot core, which now relieved of its outer burden, simply cools. If the mass is below the Chandrasekhar limiting mass for carbon white dwarfs, the star continues to cool to the virtually immortal state of a white dwarf.

***Structure and Evolution of White Dwarfs*** We have already discussed much that is relevant to the description of this abundant stellar component of the galaxy, and we will return to the subject in Section 6.4. In Section 1.3 we derived the equation of state appropriate for a relativistic and a nonrelativistic degenerate gas and found them to be polytropic. In Section 2.4 we developed the mass-radius relation for polytropes in general, which provides the approximate results appropriate for white dwarfs. In Chapter 6 we will see how the relativistic equation of state and the theory of general relativity lead to an upper limit of the mass that one can expect to find for white dwarfs. However, some description of the white dwarfs formed by the evolution of low-mass stars and their subsequent fate is appropriate.

There are basically two approaches to the theory of white dwarfs. The first is to observe that a relativistically degenerate gas will behave as a polytrope and to explore the implications of that result. The second is to investigate the detailed physics that specifies the equation of state and to create models based on the results. Cox and Giuli[8] and references therein provide an excellent example of the latter. Our approach will be much nearer the former.

The ejection of a planetary nebula during the later phases of the evolution of a low-mass star leaves a hot degenerate core of carbon and oxygen exposed to the interstellar medium. While such a core may range in mass from about 0.1 of a solar mass to more than a solar mass, its future will be remarkably independent of its mass. While the actual run of the state variables will pass through regions of degeneracy through partial degeneracy to a nondegenerate surface layer, the basic properties of the star can be understood by treating the stars as polytropes.

From observation we know that the white dwarf remains of stellar evolution are about a solar mass confined to a volume of planetary dimensions and thus will have a density of the order of $\rho = 10^6$ g/cm$^3$. If we assume that the gas is fully ionized, then the typical energy of an electron will be about 0.1 MeV for a fully degenerate gas. If the stellar core were at a temperature of $10^7$EK, the typical ion would have an energy of about 1 keV. Since energy densities are like pressures, even if the number densities were the same for the electrons and ions, the pressure of the electrons would dominate. In fact, since the typical ion produces many electrons, the dominance of the electron pressure, is even greater. Thus the structure will be largely determined by the electron pressure and the ions may be largely ignored. However, Hamada and Salpeter[9] have shown that at densities around $10^8$ g/cm$^3$ the Fermi energy of the electron "sea" becomes so high that inverse beta decay becomes likely and some of the electrons disappear into the protons of the nuclei, causing the limiting mass to be somewhat reduced over what would be expected for a purely degenerate gas. Also, the thermal energy of the ions is lost, permitting the star to shine.

Since these stars are largely degenerate, most of the momentum states in phase space are full, and an electron that is perturbed from its place in phase space has to travel quite a distance before it can find an empty place. This implies that the mean free path of electrons will be very long in spite of the high densities. Such electrons play essentially the same role as the conduction electrons of a conductor so that the electrical and thermal conductivity will be very high in a degenerate gas. This crowding of the states in phase space also results in the reduction in radiative opacity since it is difficult for a photon to move an electron from one state to another. As a result, it is very difficult for temperature gradients to exist within a fully degenerate configuration. However, in the outer regions of the white dwarf where the gas becomes partially degenerate, the opacity rapidly rises and the conductivity drops, giving rise to a steep temperature gradient with the result that the energy flow to the surface is seriously impeded. Thus Aller has likened a white dwarf to a metal ball wrapped in an insulating blanket. Since the structure of a polytrope is stable and independent of the temperature, the evolutionary history of a white dwarf largely revolves on the details of its cooling.

In 1952, Leon Mestel[10] took basically this classical approach to the cooling of white dwarfs and found that the cooling curve [dln(L)/dln(t)] was approximately constant and independent of time. Iben and Tutukov[11], using a much more detailed analysis and equation of state, found virtually the same result which they regarded as occurring through a series of accidents. Their results give [dln(L)/dln(t) ≈ (-1.4, -1.6)] for $5 < \log(t_{yrs.}) < 9.4$. It is true that a considerable number of effects complicate the simple picture of a polytrope wrapped in a blanket.

For example, while we may neglect the ions for an excellent approximation of the description of the white dwarf structure, the contribution of the ion pressure will make the star slightly larger than one would expect from the polytropic approximation. Because of the extreme concentration of the star, a small contraction produces a considerable release of gravitational energy, which is then to be radiated away. This extends the cooling time significantly over that which would be expected simply for a cooling polytrope. Toward the end of the cooling curve a series of odd things happen to the equation of state for the white dwarf. As the interior regions cool, they undergo a series of phase transitions first to a liquid state and then to a crystalline phase. Each of these transitions results in a "*heat of liquefaction or crystallization*" being released and increasing the luminosity temporarily. Problems of the final cooling remain in the understanding of the low-temperature high-density opacities that will determine the flow of radiation in this final descent of the white dwarf to a cool cinder, called a *black dwarf* in thermal equilibrium with the ambient radiation field of the galaxy. The question is of considerable interest since such objects could be detected only by their gravitational effect and could bear on the question of the "missing mass".

## b Evolution away from the Upper Main Sequence

The evolution of the more massive stars that inhabit the upper main sequence is driven by the same processes that govern the evolution of the lower main sequence, namely, the exhaustion of hydrogen fuel. However, the processes are quite different. The exhaustion of the convective core leads to the production of a helium center, as in the lower main sequence stars, but now the core will have to contend with the Chandrasekhar-Schönberg limit.

***Nature of the Massive Helium Core*** In massive stars, as the hydrogen is depleted in the helium core, the temperature rises rapidly, to produce the energy necessary to accommodate the demands of stellar structure. For stars with masses greater than about 7 $M_\odot$, the resulting helium core will be greater than the Chandrasekhar-Schönberg limit; and to make up for the energy deficit caused by the failing hydrogen burning, the core will have to contract. Since the contraction must maintain a temperature gradient, the contraction will proceed rather faster than would be expected for an isothermal core. However, the steep temperature gradient established by the terminal phases of core hydrogen burning will be relaxed because the energy generated by gravitational contraction will not be as centrally concentrated as it was from hydrogen burning. This drop in temperature gradient will cause convection to cease, yielding a core in radiative equilibrium supplying the required stellar energy by contraction.

The contraction of virtually any polytrope will result in an increase in internal temperature. This is really a consequence of the Virial theorem. However, the cessation of hydrogen burning in the core and the resultant decrease in the temperature gradient imply a change in the overall structure of the core, and thus the polytropic analogy is somewhat strained. The decline in the temperature gradient actually implies that the core will suffer a reduction of its internal energy while the boundary temperature increases. This loss of internal energy goes into the support of the outer envelope. Thus, both this energy and the energy generated by contraction are available for the support of the outer layers of the star.

The increase in the boundary temperature of the core will result in a slow expansion and cooling of the radiative envelope for the same reason described for low-mass stars. After a suitable rise in the boundary temperature, hydrogen is reignited in a shell surrounding the helium core. This results in a marked change in the temperature gradient of the hydrogen-burning shell at the core boundary. The localization of the energy production in such a small region steepens the temperature gradient to the point where the temperature gradient exceeds the adiabatic gradient, driving the entire envelope into convection. The outer envelope now rapidly transfers the energy to the surface, which again becomes the limiting barrier to its escape. The star moves rapidly toward the giant branch as a star with a helium core surrounded by a hydrogen-burning shell and covered with a deep convective envelope. This envelope is so deep that it reaches into the region where nuclear processing has taken place, dredging up some of this material to the surface. The result of this process is evident in the atmospheric spectra of some late-type supergiants.

Stars with masses less than about $5M_\odot$ will end their hydrogen burning with a helium core below that of the Chandrasekhar-Schönberg limit and will contend with a slowly contracting isothermal core right through the ignition of a hydrogen-burning shell. The future of such a star is mirrored in the behavior of lower-mass stars except that helium ignition takes place before the core becomes significantly degenerate. The result is that these stars experience no helium flash, and the transition to helium burning is orderly.

***Ignition of the Massive Helium Core*** In both cases described above, the contraction of the helium core proceeds while the hydrogen shell is burning. In the more massive stars, where the core is above the Chandrasekhar-Schönberg limit, the star must do so to maintain a temperature gradient for its own support. In the less massive case, the core grows slowly as a result of the processing of the hydrogen in the energy-generating shell. The added mass results in a slow core contraction. In both instances, the decreasing density in the hydrogen-burning shell necessitates a rise in the temperature required for energy generation. After the increasing temperature gradient caused by this increasing shell temperature has forced the envelope to become fully convective, the convective envelope continues to expand, for the energy escape is again limited by the radiative efficiency of the photosphere.

Eventually the central temperature reaches $10^8$ K, and helium ignition takes place. The ignition has a dramatic effect on the core but does not exhibit the explosive nature of the helium flash. The core undergoes a rapid expansion and, because of the huge temperature dependence of the triple-a process, becomes unstable to convection. This produces an expanded convective helium core surrounded by a hydrogen-burning shell. This shell supplies more than 90 percent of the energy required to maintain the luminosity. The rapid core expansion is accompanied by a contraction of the outer envelope with a corresponding increase in the surface temperature. The star moves off to the left in the H-R diagram, maintaining about the same total luminosity. The hydrogen burning shell continues to supply the majority of the energy throughout the helium burning phase which proceeds in much the same manner as the main sequence core hydrogen-burning phase, but on a much shorter time scale.

***Terminal Phases of Evolution of Massive Stars*** The end phase of the evolution of massive stars is still somewhat murky and the subject of active research. Initially, the helium burning continues in the core until the core becomes largely carbon. At the point of helium exhaustion, the core again gravitationally contracts, rising in temperature, until helium is ignited in a shell source around the core. For stars with a mass between 3 and 7 solar masses, there is some evidence that the carbon core which develops is degenerate and that ignition, when it occurs, occurs explosively, perhaps producing a supernova. For stars of more than 10 solar masses, carbon burning can take place in a nonviolent manner, producing cores of neon, oxygen, and finally silicon, each surrounded by a shell source of the previous core material which continues to provide some energy to the star. The results of silicon burning yield elements of the iron group for which further nuclear reactions are endothermic, and so this burning will not only fail to contribute energy for the support of the star but also rob it of energy. In addition, the densities become high enough that the electrons are forced into the protons of the nuclei by means of inverse b decay. This effect has been called *neutronization* of the core. Both mechanisms produce a significant number of neutrinos which also do not take part in the support of the star against gravity and can be viewed as "cooling" mechanisms for the core.

This rapid cooling precipitates a rapid collapse of the core followed by the entire star. The in-fall velocity soon exceeds the speed of sound, resulting in the formation of a shock wave, and interior densities may become large enough for the material to become opaque to neutrinos. Under some conditions the endothermic nuclear reactions may bring about the disintegration of the iron-group elements into $^4$He. Which process dominates for stars of which mass is not at all clear. The shock wave formed by the in-fall may "bounce", or the increased neutrino opacity may provide sufficient energy and momentum to ensure that a large fraction of the star will be blown into the interstellar medium.

The remnants, if any, could be a neutron star or a black hole. Although the details of the terminal phases of massive stars remain somewhat unclear, it is virtually certain that these phases are likely to end with the production of a supernova and the subsequent enrichment of the interstellar medium by heavy elements.

### c The Effect of Mass-loss on the Evolution of Stars

Throughout our discussion of stellar evolution we have assumed that the mass of the evolving star remains essentially constant. Certainly there is a reduction in mass from the nuclear production of energy, but this can never exceed 0.7 percent and therefore can be safely neglected. However, as we shall see in section 16.3, stars may exhibit rather large winds emanating from their atmospheres. In some cases these winds may result in mass loss rates exceeding $10^{-5}M_\odot$ per year and so could lead to a significant reduction in the mass of the star during its nuclear lifetime.

Ever since the observation by Armin Deutch[12] that the red supergiant $\alpha$ Herculis was losing mass faster than about $10^{-8}$ $M_\odot$ per year, interest in the effects of mass loss on the evolution of a star has been high. Generally one would expect that a star slowly losing mass would follow the classical evolutionary track represented by models of stars having successively lower and lower mass. Thus for a main sequence star that is losing mass at a significant rate the evolutionary track on the HR diagram would not rise like the constant mass models, but move steadily to the right and perhaps down until the asymptotic giant branch is reached. In addition, the lifetime would be significantly enhanced as a result of the reduced stellar mass. Such were the conclusion reached by Massevitch[13] and collaboraters[14] in the late 1950's. However, the evolutionary tracks of 1-2$M_\odot$ failed to fit the HR diagrams of globular clusters and in the absence of evidence for mass loss from main sequence stars, their work was largely ignored. However, during the 1960s and 70s it became clear that virtually all early type stars exhibited significant stellar winds and it was likely that their evolution from the main sequence was affected[13,14]. In general, the more massive the star, the greater the fractional mass loss rate will be. This would explain why stars in the range of a few solar masses where unaffected and the constant mass models fit the globular cluster HR diagrams relatively well. However, from studies of the ratio of blue to red supergiants in the Milky Way and other galaxies Humphreys and Davidson[15] concluded that stars more massive than 50$M_\odot$ never made it to the red supergiant phase but remained confined to the left hand side of the HR diagram throughout their lives. Lamers[16] found that this did not appear to be the case for stars in our galaxy concluding that a 100$M_\odot$ could only loose 15 percent of its mass during it main sequence lifetime.

The impact of such mass loss on the subsequent evolution of the star seems to depend on an accurate knowledge of the mass loss rate during the the evolution which in turn rests on the specifics of the origin of stellar winds. Using an empirically inspired mass loss rate, Brunish and Truran[17] found that mass loss affected the evolution of stars less than 30$M_\odot$ more than the massive stars. However, Sreenivasan and Wilson[18,19] found including rotation and a theoretically motivated origin for the mass loss rate resulted in a much more complicated evolutionary history. By adjusting the amount of rotation present in the initial star they are able to match most of the observations. However, it is fair to say that as of the present much remains poorly understood about the specific role played by mass loss in the massive stars. It also appears that a proper understanding will require models that correctly couple the atmosphere to the interior and include rotation in a physically self consistent way.

## 5.5 Summary and Recapitulation

In this chapter we have sketched the evolution of normal stars from their contraction out of the interstellar medium to their probable fate. We have not discussed many topics and details which are important to the detailed understanding of stellar evolution and some important problems remain unsolved. For the evolution of individual stars, an area of singular importance that was acknowledged only in passing concerns the origin of the elements. The production of the elements through nuculosynthesis was suggested by Burbidge, Burbidge, Fowler and Hoyle[20] and the early view of the important processes are reviewed by Bashkin$^2{}_1$. The manifestation of these elements in the outer layers of the star and the internal processes by which they got there are covered by Wallerstein[22]. In addition, we have said nothing about the fascinating topic of the evolution of close binary stars where the futures of the components are linked through the process of mass exchange. We have said nothing about the mass loss from massive stars that may alter the evolution of these stars. Nor have we touched on the tricky processes by which a white dwarf cools off. We also avoided the effects of rotation and magnetic fields on the evolution of stars along with the details of the dynamic collapse of stars, and these should be regarded as fertile areas for research. However, we did delineate major events in the lives of normal stars. Specifically, we used the efficiency of energy transport, the temperature sensitivity of the nuclear reactions, and the radiative ability of the photosphere to indicate the probable direction that the evolution of stars will take. Simple arguments of efficiency lead to a remarkably accurate description of the early phases of stellar evolution to the main sequence. Post main sequence evolution is made more complicated by the zonal nature of the star, complications to the equation of state, and the existence of multiple energy sources. Nevertheless, we can see the basic conservation laws of physics at work during the latter phases of stellar evolution and can get a feel for the important processes at work. We close this discussion with another view of the interplay between the core and outer envelope along with a detailed look at the evolution of a $5M_\odot$ star.

### a Core Contraction - Envelope Expansion: Simple Reasons

For years there has been some debate over why the envelope of an evolving star expands when the core contracts, for many people find the result counterintuitive. A number of explanations have been suggested and objections have been made to almost all, of being simplistic or incomplete. Some have regarded the question as being so complicated that it is not useful to search for a single cause, and in response to the question of envelope expansion they simply say, "It happens because my computer tells me it does." This is no answer at all, for it offers no insight into the physical phenomena that result in the particular behavior exhibited by the star. Certainly the physical situation which leads to the expansion of the envelope during core contraction is complicated and simple answers, in some sense, will always be incomplete. However, we should make the effort to identify the important processes at work which dominate the result.

It would be useful if we could clarify the question a little. A star goes through several different phases as it evolves from the main sequence to the giant branch of the H-R diagram, and all result in an expansion of the envelope accompanying some contraction of the core. However, the structure of the core and that of the envelope differ widely in these various phases as do the magnitude and time scale for the resulting core contraction-envelope expansion. We attempted to make plausible the expansion of the convective envelope which accompanies the temperature increase of the hydrogen-burning shell, resulting from the contraction of the radiative helium-rich core, by appealing to the behavior of convective polytropes. Although such envelopes will not be the complete polytropes of the Hayashi tracks, it seems reasonable that the stars will approach the tracks in their general behavior, for the same principles that result in the Hayashi tracks are operative in the expansion of the convective envelope. However, in envelope expansion, the processes are reversed and less perfectly followed, since only the envelope is involved. The general expansion of the radiative zone overlying the helium core which initiates the departure of the star from the main sequence cannot be described in the same manner. The expansion would be represented by a partial polytrope having an index that varies in time. Indeed, for the lower main sequence stars, that zone is surrounded by a convective region that, together with the radiative zone, makes up the envelope outside the hydrogen-burning shell. This compound envelope undergoes the expansion.

With such a great variety of situations leading to envelope expansion, does it even make sense to look for a common cause? If we found one, it would be necessarily vague about details since it must apply in a wide variety of circumstances. Should this common cause exist, it must result from a very general principle in order for it to apply in these many diverse circumstances. Let us consider two very general principles to see if they can provide a qualitative indication as to how the entire star will behave should the central regions, embodying the majority of the mass, contract. First, the conservation of energy will require that the total energy of the star be written as

$$E = \langle \Omega \rangle + \langle U \rangle - \int_0^t L dt + \int_V \int_0^t \varepsilon dt \, dV \tag{5.5.1}$$

For any time scale that is less than the Kelvin-Helmholtz time, the magnitude of the integrals will be less than either the gravitational or internal energy, because the Kelvin-Helmholtz time is essentially the time required for the star's luminosity to lose an amount of energy equal to the internal energy. Since the integrals appear with opposite sign and are approximately equal, and since we have included all sources of energy available to the star explicitly, we may write

$$E . \langle\Omega\rangle + \langle U\rangle . \text{constant} \tag{5.5.2}$$

The contractions and expansions of interest occur on time scales very much longer than the dynamic time scale, so we can be certain that the time-averaged form of the Virial theorem will apply. Thus

$$\langle\Omega\rangle + 2\langle U\rangle = 0 \tag{5.5.3}$$

The combination of equations (5.5.2) and (5.5.3) requires that the gravitational and potential energy, separately, be constant. Now as long as we average over a dynamical time, equation (5.5.3) will be valid, while equation (5.5.2) becomes more exact for shorter times t since the integrals of equation (5.5.1) will contribute less. Thus we may drop the averages and expect that for any time greater than the dynamical time but shorter than the Kelvin-Helmholtz time

$$\Omega . \text{constant} . \Omega_c + \Omega_e \tag{5.5.4}$$

Now, for upper main sequence stars, the mass of the core substantially exceeds that of the envelope,

$$\Omega \,.\, GM_c^2/R_c + GM_cM_e/R_* \tag{5.5.5}$$

If, for simplicity, we further hold the masses of the core and envelope constant during the core contraction, we have

$$\frac{dR_*}{dR_c} \approx -\left(\frac{M_c}{M_e}\right)\left(\frac{R_*}{R_c}\right)^2 \ll -1 \tag{5.5.6}$$

The sign of equation (5.5.6) indicates that we should expect the observed radius of the star to increase for any decrease in the core radius, and the magnitude of the right-hand side implies that a very large amplification of the change in core size would be seen in the stellar radius. One can argue that the assumptions are only approximately true or that the time scales involved occasionally approach the Kelvin-Helmholtz time, but that will affect only the degree of the change, not the sign. Indeed, detailed evolutionary model calculations throughout the period of evolution from the main sequence to the giant branch indicate that the total gravitational and internal energy is indeed constant to about 10 percent. The accuracy for shorter times is considerably better. For lower main sequence stars, the mass of the core is less than that of the envelope. Nevertheless, a result similar to equation (5.5.6), with the same sign, is obtained although the magnitude of the derivative is not as large.

The nature of this argument is so general that we may expect any action of the core to be oppositely reflected in the behavior of the envelope regardless of the relative structure. Thus, we can understand the global response of the star to the initial contraction of the core when the overlying layers are in radiative equilibrium as well as the subsequent rapid expansion to and up the giant branch when the outer envelope is fully convective. In addition, contraction of the stellar envelope following the core expansion accompanying the helium flash, which leads to its position on the horizontal branch, can also be qualitatively understood. In general, whenever the core contracts, we may expect the envelope to expand and vice versa. Detailed model calculations confirm that this is the case.

**Table 5.1 History of a $5M_\odot$ Star**

| Point | Duration | Elapsed Time | |
|---|---|---|---|
| *Location* | *yr* | *yr* | Primary Physical Activity |
| (1–2) | $6.4 \times 10^7$ | $6.40 \times 10^7$ | H burning core |
| (2–3) | $2.2 \times 10^6$ | $6.62 \times 10^7$ | Core exhaustion and contraction |
| (3–4) | $1.4 \times 10^5$ | $6.63 \times 10^7$ | Establishment of hydrogen-burning shell |
| (4–5) | $1.2 \times 10^6$ | $6.75 \times 10^7$ | H shell thickens |
| (5–6) | $8.0 \times 10^5$ | $6.83 \times 10^7$ | H exhaustion, envelope expansion to convection |
| (6–7) | $5.0 \times 10^5$ | $6.88 \times 10^7$ | Core contraction, envelope expansion |
| (7–8) | $6.0 \times 10^6$ | $7.48 \times 10^7$ | He ignition and burn, envelope contraction, and core expansion |
| (8–9) | $1.0 \times 10^7$ | $8.48 \times 10^7$ | Primary He burning phase |
| (9–10) | $1.0 \times 10^6$ | $8.58 \times 10^7$ | He core grows, envelope expands |
| (10–11) | $< 10^5$ | $8.58 \times 10^7$ | Core contraction, He shell ignition |
| (11–12) | $< 10^4$ | $8.58 \times 10^7$ | He exhaustion before C ignition |

## b *Calculated Evolution of a 5 $M_\odot$ star*

In this final section we look at the specific track on the H-R diagram made by a $5M_\odot$ star as determined by models made by Icko Iben. This is best presented in the form of a figure and is therefore shown in Figure 5.2 above. Similar calculations have been done for representative stellar masses all along the main sequence so the evolutionary tracks of all stars on the main sequence are well known. The basic nature of the theory of stellar evolution can be confirmed by comparing the location of a collection of stars of differing mass but similar physical age with the H-R diagrams of clusters of stars formed about the same time. A reasonable picture is obtained for a large variety of clusters with widely ranging ages. It would be presumptuous to attribute this picture to chance. While much remains to be done to illuminate the details of certain aspects of the theory of stellar evolution, the basic picture seems secure.

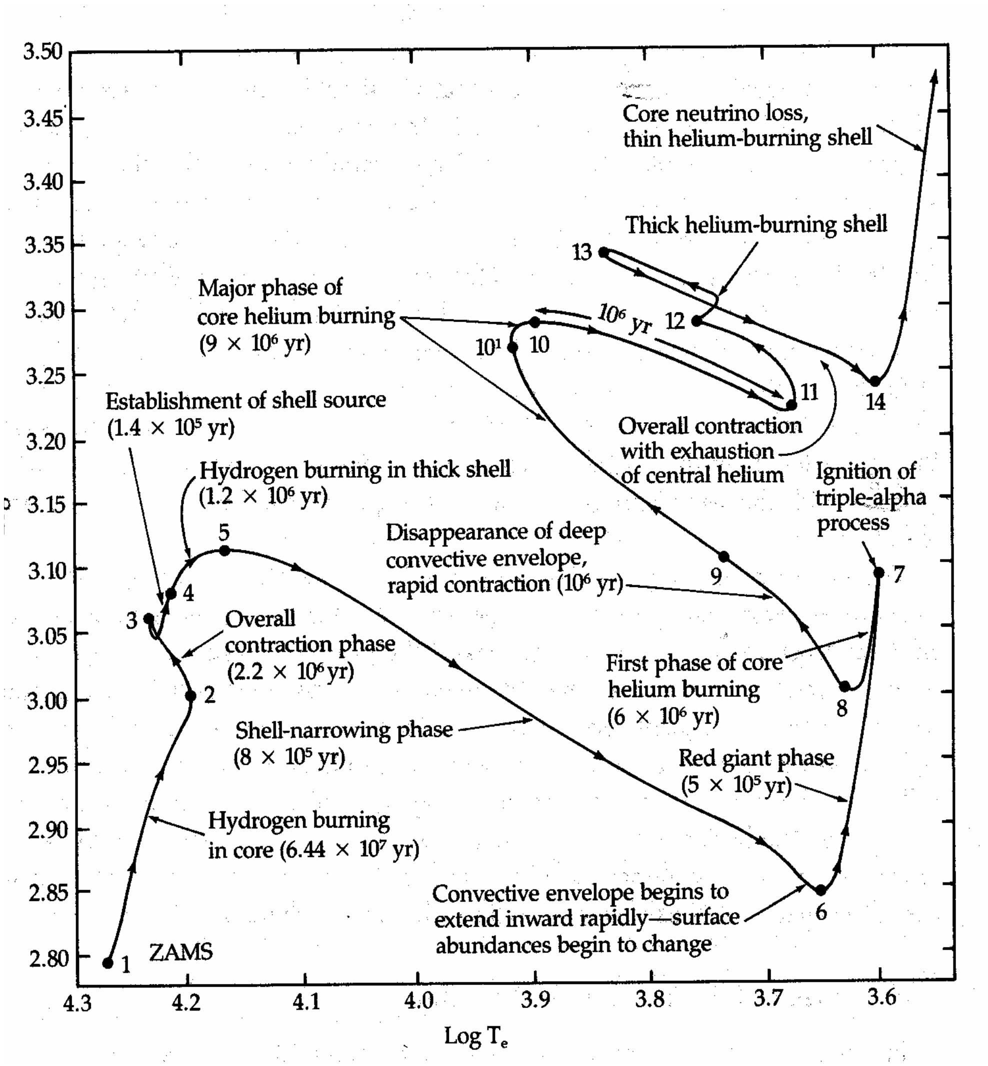


**Figure 5.2** delineates the evolution of a $5M_\odot$ star from its arrival on the main sequence through its demise at the onset on carbon burning[23]. The labeled points are points of interest discussed in the chapter and their duration, place in the stellar lifetime, and the significant physical process taking place are given in Table 5.1.

## Problems

1. Find the fraction by mass and radius inside of which 20 percent, 50 percent, and 99 percent of the sun's energy is generated. Compare the results with a star of the same chemical composition but with 10 times the mass.

2. Determine the mass for which stars with the chemical composition of the sun derive equal amounts of energy from the CNO and p-p Cycles.

3. Determine the relative importance of free-free and bound-free absorption and electron scattering as opacity sources in the sun.
4. Calculate the evolutionary tracks for a $1M_\odot$ star and $10M_\odot$ star.
5. Choose a representative set of models from the evolutionary calculations in Problem 4, (a) Calculate the moment of inertia, gravitational and internal energies of the core and envelope, and the total energy of the star (b) Determine the extent to which the conditions in Section 5.5a are met during the evolution of the star.
6. Compute the Henyey track for a $1M_\odot$ star, and compare it with that of a polytrope of index $n$ = 3. Would you recommend that the comparison be made with a polytrope of some different index? If so, why?
7. Compute the evolutionary track for the sun from early on the Hayashi track as far as you can. At what point do you feel the models no longer represent the actual future of the sun, and why?
8. Discuss the evolution of a $5M_\odot$ star as it leaves the main sequence. Detail specifically the conditions that exist immediately before and after the onset of hydrogen-shell burning.
9. Consider a star composed of an isothermal helium core and a convective hydrogen envelope. Suppose that the mass in the core remains constant with time but that the core contracts. By constructing a model of appropriate polytropes, show what happens to the envelope and comment on the external appearance of the star.

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or

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