# 7

# Structure of Distorted Stars

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Throughout this book we have assumed that stars are spherical. This reduces the problem of stellar structure to one dimension, greatly simplifying its description. Unfortunately, many stars are not spherical, but are distorted by their own rotation or the presence of a nearby companion. Not only does this add geometric complications to the mathematical representation of the equations of stellar structure, but also new physical phenomena, such as global circulation currents, may result. Major problems are created for the observational comparison with theory in that the appearance of a star will now depend on its orientation with respect to the observer. Some quantities, such as the total luminosity and the stellar effective temperature, are no longer accessible to observation. With these problems in mind, we consider some approaches to developing a theoretical framework for the structure of distorted stars.

The removal of spherical symmetry, by increasing the number of dimensions required for the description of the star's structure, will considerably increase the number of equations to be solved to obtain that structure. Rather than develop those equations in detail, we indicate how they are obtained and the basic procedures for their solution. We consider only those cases that exhibit axial symmetry so that the number of dimensions is increased by 1. This is sufficient to illustrate most problems generated by distortion without raising the complexity to an unacceptable level. It also provides a framework for the description of a significant number of additional stars.

## 7.1 Classical Distortion: The Structure Equations

The loss of spherical symmetry will change the familiar equations of stellar structure to vector form. Before developing the specific equations for axial distortion, let us consider the general form of these equations. In Chapter 6 we compared the relativistic equations of stellar structure to the classical spherical equations. In a similar manner, let us begin our discussion of distortion with a comparison of the classical spherical equations with their counterparts for distorted stars.

### a A Comparison of Structure Equations

Below is a summary of the equations of stellar structure for spherically symmetric stars and stars which suffer a general distortion.

| | Spherical | | Nonspherical |
|---|---|---|---|
| (a) | $\dfrac{dM(r)}{dr} = 4\pi r^2 \rho$ | mass conservation | $\nabla^2 \Omega = 4\pi G \rho$ Poisson's eq. |
| (b) | $\dfrac{dL(r)}{dr} = 4\pi r^2 \rho \epsilon$ | energy conservation | $\nabla \cdot \vec{F} = \rho\epsilon - T\dfrac{\partial S}{\partial t}$ |
| (c) | $\dfrac{dT(r)}{dr} = -\dfrac{3\bar{\kappa}\rho L(r)}{16\pi a c T^3 r^2}$ | radiative equilibrium | $\nabla T = -\dfrac{3\bar{\kappa}\rho}{4acT^3}\vec{F}$ |
| (d) | $\dfrac{dP(r)}{dr} = -\dfrac{GM(r)\rho}{r^2}$ | hydrostatic equilibrium | $\nabla P = -\rho\nabla\Omega + \rho\vec{D}$ |
| | $\kappa = \kappa(P, T, \rho)$ | opacity | $\kappa = \kappa(P, T, \rho)$ |
| (e) | $\epsilon = \epsilon(P, T, \rho)$ | energy generation | $\epsilon = \epsilon(P, T, \rho)$ |
| | $P = P(T, \rho, \mu)$ | equation of state | $P = P(T, \rho, \mu)$ |

(7.1.1)

The variable $M$(r) that is so useful for spherical structure is replaced by the potential, given here as the gravitational potential. In principle, the potential could contain a contribution from other physical phenomena such as magnetism or rotation. Poisson's equation is a second-order partial differential equation and replaces the first-order total differential equation for spherical structure. So the price we pay for the loss of spherical symmetry is immediately obvious. While the conservation of energy equation remains a scalar equation, as it should, it now involves a vector quantity, the radiative flux, and an additional term that anticipates some results from later in the chapter. The quantity S is the entropy of the gas, and in Chapter 4 [see equation (4.6.9)] we saw that this term had to be included when the models were changing rapidly in time. In this case, the term is required to describe the flow of energy due to mass motions resulting from the distortion itself. Both radiative and hydrostatic equilibrium become vector equations where we have explicitly indicated the presence of a perturbing force by the vector $\vec{D}$ which, should it be derivable from a potential, could be included directly in the potential. This perturbing force is assumed to be known. The quantities such as $\kappa$ and $\varepsilon$, which depend on the local microphysics, presumably will not be directly affected by the presence of a macroscopic perturbing force. A possible exception could be the case of distortion by a magnetic field where the local field would contribute to the total pressure and in extreme cases, could affect the opacity.

### b Structure Equations for Cylindrical Symmetry

To minimize the complexity, we consider those cases resulting in the loss of only one symmetry coordinate, and we deal with those systems exhibiting axial symmetry. This is clearly appropriate for rapidly rotating stars as well as stars distorted by the presence of a companion. In addition, we shall see that it also is appropriate for the distortion introduced by an ordered magnetic field that itself exhibits axial symmetry.

To specifically see the effects that result from a distortion force, we have to express that force in some appropriate coordinate system. The distortion force was represented in the structure equations, (7.1.1), by the vector $\vec{D}$ in the equation of hydrostatic equilibrium. For axial symmetry, cylindrical and spherical polar coordinates both form suitable coordinate systems for this description (see Figure 7.1). We express the components of the perturbing force in terms of Legendre polynomials of the polar angle $\theta$. Once the perturbing force has been characterized, we shall indicate, in the next section, how the solution of the structure equations proceeds.

The Legendre polynomials form an orthogonal set of polynomials over a finite, defined range. Specifically, let

$$\mu = \mathrm{Cos}\,\theta \tag{7.1.2}$$

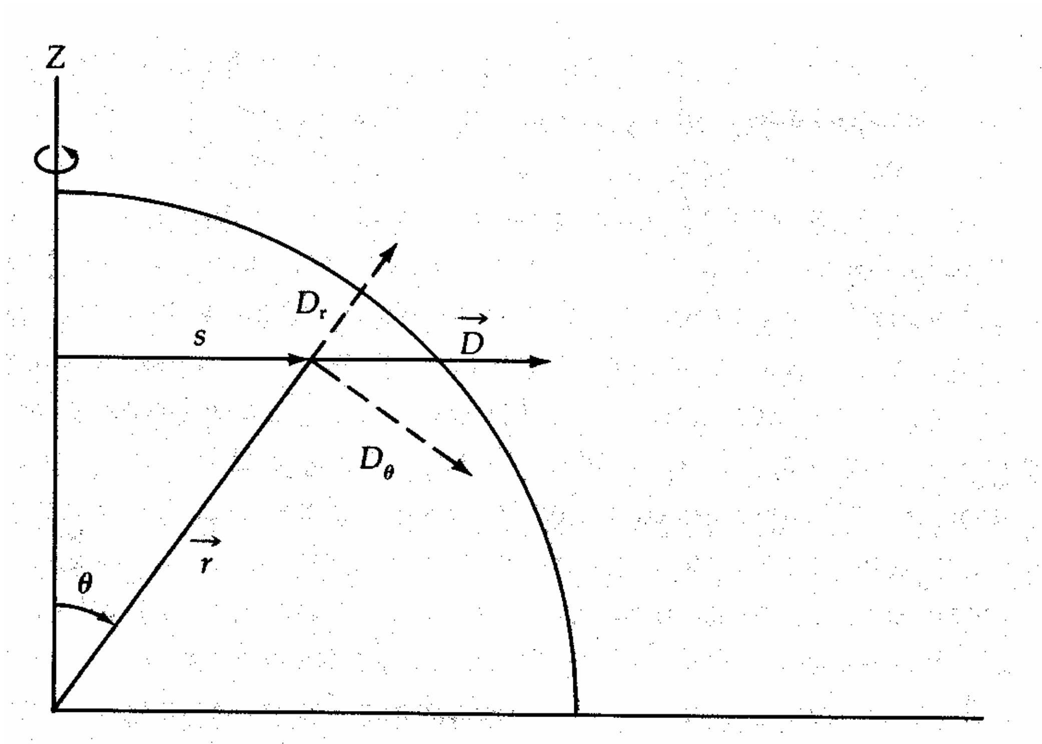


Figure 7.1 shows suitable coordinate systems to describe an axially symmetric perturbing force $\vec{D}$ with components $D_r$ and $D_\theta$. We have chosen to illustrate rotational distortion so that Z represents the spin axis. However, we could have illustrated gravitational distortion by an external object in which case Z would lie along the line of centers of the system and $\vec{D}$ would point to the center of the other object.

Then the Legendre polynomials form an orthonormal set in the interval $-1 \le \mu \le +1$ subject to the normalization condition

$$\int_{-1}^{+1} P_n(\mu)P_m(\mu)\,d\mu = \delta_{m,n}\left(\frac{2}{2n+1}\right) \tag{7.1.3}$$

Here $\delta_{m,n}$ is the Kronecker delta which is 1 if m = n and 0 otherwise. Various members of the set of Legendre polynomials can be generated from the recursion relation

$$P_{m+1}(\mu) = \frac{2m+1}{m+1}\mu P_m(\mu) - \frac{m}{m+1}P_{m-1}(\mu) \tag{7.1.4}$$

where the first three members of the set are

$$P_0(\mu) = 1$$
$$P_1(\mu) = \mu = \text{Cos}\,\theta$$
$$P_2(\mu) = \frac{3}{2}\mu^2 - \frac{1}{2} = 1 - \frac{3}{2}\text{Sin}^2\theta \tag{7.1.5}$$

Before we can specify the effects of the perturbing force in detail, we must indicate its nature. So let us turn to some simple examples of distorting forces and their effects on the structure equations.

***Rigid Rotation*** For our first example, we consider the case where the star is rotating as a rigid body. This yields a simple expression for the magnitude of the distorting force produced by the local centripetal acceleration, which is

$$|D| = \omega^2 r\,\text{Sin}\,\theta \tag{7.1.6}$$

where ω is the angular velocity of the star and is assumed to be constant. The components of the acceleration are then

$$D_r = \omega^2 r\,\text{Sin}^2\theta \qquad D_\theta = \omega^2 r\,\text{Sin}\,\theta\,\text{Cos}\,\theta \tag{7.1.7}$$

which can be expressed in terms of Legendre polynomials and their derivatives as

$$D_r = \frac{2}{3}\omega^2 r[1 - P_2(\mu)] \qquad D_\theta = -\frac{1}{3}\omega^2 r\frac{\partial P_2(\mu)}{\partial\theta} \tag{7.1.8}$$

Due to the axial symmetry, $D_\phi = 0$ and it is a simple matter to show that the curl of D, $\nabla \times \vec{D}$, is 0 so that $\vec{D}$ is derivable from a scalar potential by

$$\vec{D} = -\nabla\Lambda \tag{7.1.9}$$

where

$$\Lambda = -\tfrac{1}{2}\omega^2 r^2\,\text{Sin}^2\theta \tag{7.1.10}$$

Thus, the components of the perturbing force and the rotational potential can be expressed in terms of the Legendre polynomials as

$$D_r = A(r) + B(r)P_2(\mu)$$
$$D_\theta = C(r)\frac{\partial P_2(\mu)}{\partial\theta}$$
$$\Lambda = Q_0(r) + Q_1(r)P_2(\mu) \tag{7.1.11}$$

Although the above relations are correct for ω = constant, it is worth considering the functional dependence of ω for which it is true in general. Consider the nature of centripetal acceleration in a cylindrical coordinate system where the radial coordinate is denoted by s. The components of $\vec{D}$ are

$$D_z = D_\phi = 0\ , \quad D_s = \omega^2 s \tag{7.1.12}$$

In order for the rotational force to be derivable from a scalar potential, its curl must be zero. The cylindrical components of the curl are

$$(\nabla\times\vec{D})_s = \frac{1}{s}\frac{\partial D_z}{\partial\phi} - \frac{\partial D_\phi}{\partial s} = 0$$
$$(\nabla\times\vec{D})_z = \frac{\partial D_s}{\partial\phi} = 0$$
$$(\nabla\times\vec{D})_\phi = \frac{\partial D_s}{\partial z} - \frac{\partial D_z}{\partial s} = \frac{\partial(\omega^2 s)}{\partial z} \tag{7.1.13}$$

The radial component is identically zero, so we may suspect that if the object exhibits axial symmetry, ω cannot be a function of $\phi$. In this case, the z component of the curl would also be zero. Thus, the condition that the rotational force be derivable from a scalar potential boils down to the $\phi$ component of the curl being zero, so that

$$\frac{\partial(s\omega^2)}{\partial z} = 0 \tag{7.1.14}$$

Thus,

$$\omega \neq \omega(z) \tag{7.1.15}$$

so the angular velocity must be constant on cylinders. It can be shown, that if the perturbing force is not derivable from a potential, then no equilibrium solution of the structure equations exists. This is sometimes called the *Taylor-Proudman theorem*[1] and it basically guarantees that if the star has reached an equilibrium angular momentum distribution, the angular velocity will be constant on cylinders.

***Gravitational Distortion by an External Point Mass*** Now let us return to the spherical polar coordinates that we used to obtain the components of the rotational force. The force will be directed toward an external point mass located along the z axis at a distance d from the center of the star (see Figure 7.2).

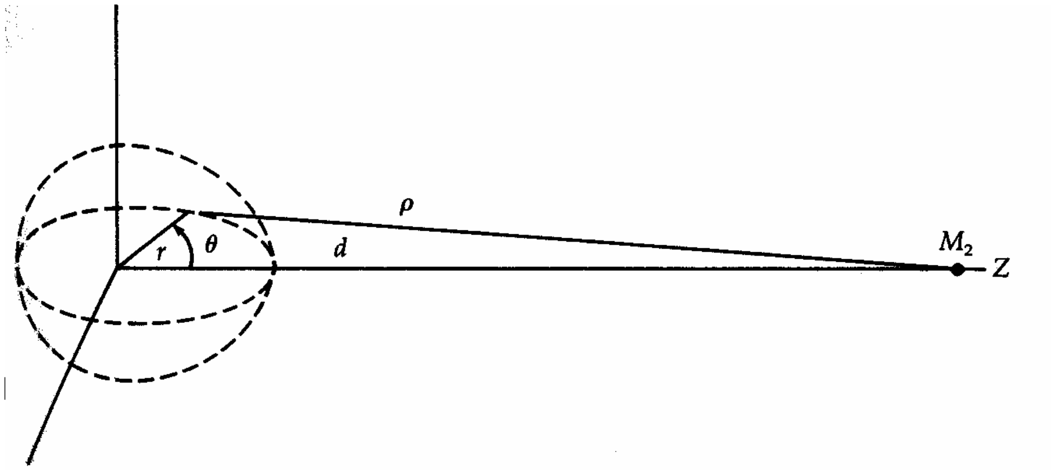


Figure 7.2 shows the type of distortion to be expected from the presence of a companion such as would be found in a close binary system. For simplification, the rotational distortion is considered to be negligible. The distance from any point in the star to the perturbing mass is denoted by $\rho$.

Now the perturbing potential of the point mass $M_2$ is

$$\Omega_2 = \frac{GM_2}{\rho} \tag{7.1.16}$$

where

$$\rho^2 = d^2 + r^2 - 2rd\,\mathrm{Cos}\,\theta = d^2\left[1 + \left(\frac{r}{d}\right)^2 - 2\frac{r}{d}\,\mathrm{Cos}\,\theta\right] \tag{7.1.17}$$

So the potential can be written in terms of our coordinates and the stellar separation as

$$\Omega_2 = \frac{GM_2}{d}\left[1 - 2\frac{r}{d}\,\mathrm{Cos}\,\theta + \left(\frac{r}{d}\right)^2\right]^{-1/2} \tag{7.1.18}$$

Equation (7.1.18) is rather non-linear in the $\theta$ coordinate so, in order to express the potential in term of Legendre polynomials, we can make use of the "generating function" (see Arfken[2] for the development of this generating function) for the Legendre polynomials,

$$(1 - 2\alpha\mu + \alpha^2)^{-1/2} = \sum_{i=0}^{\infty} P_i(\mu)\alpha^i \tag{7.1.19}$$

so the perturbing potential becomes

$$\Omega_2 = \frac{GM_2}{d} \sum_{i=0}^{\infty} \left(\frac{r}{d}\right)^i P_i(\text{Cos}\,\theta) \tag{7.1.20}$$

Since the perturbing force is conservative, we may obtain it from

$$\vec{D} = +\nabla\Omega_2 \tag{7.1.21}$$

which has components

$$D_r = \frac{GM_2}{d} \sum_{i=1}^{\infty} \left(\frac{i}{d}\right)\left(\frac{r}{d}\right)^{i-1} P_i(\text{Cos}\,\theta)$$

$$D_\theta = \frac{GM_2}{d} \sum_{i=0}^{\infty} \left(\frac{1}{r}\right)\left(\frac{r}{d}\right)^{i} \frac{\partial P_i(\text{Cos}\,\theta)}{\partial\theta}$$

$$D_\phi = 0 \tag{7.1.22}$$

So far, the only approximation that we have made is that the perturbing potential is that of a point mass. To simplify the remaining discussion, we assume that the point mass is distant compared to the size of the object so that

$$D_r = \frac{GM_2}{d}\left[\frac{\text{Cos}\,\theta}{d} + \frac{2r}{d^2} P_2(\text{Cos}\,\theta) + \cdots +\right]$$

$$D_\theta = \frac{GM_2}{d}\left[\frac{-\text{Sin}\,\theta}{d} + \frac{r}{d^2}\frac{\partial P_2(\text{Cos}\,\theta)}{\partial\theta} + \cdots +\right] \tag{7.1.23}$$

Note that the zeroth order terms of the components can be added vectorially to give

$$\vec{D}_0 = \frac{GM_2}{d^2}\hat{d} \tag{7.1.24}$$

This is just the gravitational force that is balanced by the acceleration resulting from the orbital motion of the system, and so this force can be made to vanish by going to a rotating coordinate system. In such a system, the components of the perturbing force will just be the first-order terms, so that

$$\tilde{D}_r = \frac{2GM_2 r}{d^3} P_2(\text{Cos}\,\theta) \qquad \tilde{D}_\theta = \frac{GM_2 r}{d^3}\frac{\partial P_2(\text{Cos}\,\theta)}{\partial\theta} \tag{7.1.25}$$

These components of the perturbing force have the same form as those of rotation [see equation (7.1.11)], and any method which is applicable to the solution of the structure equations for rotational distortion will also be applicable to the problem of gravitational distortion.

***Distortion Resulting from a Toroidal Magnetic Field*** Consider the Lorentz force of an internal magnetic field on the material of the star:

$$\vec{f} = \frac{\vec{j} \times \vec{B}}{c} = -(4\pi c)^{-1}[\vec{B} \times (\nabla \times \vec{B})] \tag{7.1.26}$$

The perturbing acceleration due to this force will be

$$\vec{D} = \frac{\vec{f}}{\rho} = (4\pi\rho c)^{-1}[\vec{B} \times (\nabla \times \vec{B})] \tag{7.1.27}$$

Now assume a special, but not implausible, geometry for the internal stellar magnetic field. Specifically, let us choose a toroidal field that exhibits pure axial symmetry so that

$$\vec{B} = \psi(r)(\text{Sin}\,\theta)(\hat{\phi}) \tag{7.1.28}$$

where $\psi(r)$ contains the arbitrary, but presumed known, variation of the field with the radial coordinate r. Since the field only has a $\phi$ component, the vector part of equation (7.1.27) is

$$\vec{B} \times (\nabla \times \vec{B}) = -\hat{r}[B_\phi(\nabla \times \vec{B})_\theta] + \hat{\theta}[B_\phi(\nabla \times \vec{B})_r] \tag{7.1.29}$$

The $\theta$ and r components of the curl of $\vec{B}$ are

$$\begin{aligned}(\nabla \times \vec{B})_\theta &= -r^{-1}\frac{\partial(rB_\phi)}{\partial r} = -\frac{\left[B_\phi + r\,\text{Sin}\,\theta\left(\frac{\partial\psi}{\partial r}\right)\right]}{r} \\ (\nabla \times \vec{B})_r &= (r\,\text{Sin}\,\theta)^{-1}[\psi(r)\,\text{Sin}\,\theta\,\text{Cos}\,\theta + B_\phi\,\text{Cos}\,\theta] = \frac{2\psi(r)\,\text{Cos}\,\theta}{r}\end{aligned} \tag{7.1.30}$$

which yields for the vector components of the perturbing field

$$\begin{aligned}D_r &= (4\pi\rho c)^{-1}\left[\frac{\psi^2(r)}{r} + \psi(r)\frac{\partial\psi}{\partial r}\right]\text{Sin}^2\theta = \tilde{A}(r) + \tilde{B}(r)P_2(\text{Cos}\theta) \\ D_\theta &= ((4\pi\rho c)^{-1}\frac{\psi^2(r)}{r}\frac{\partial P_2(\text{Cos}\theta)}{\partial\theta} = \tilde{C}(r)\frac{\partial P_2(\text{Cos}\theta)}{\partial\theta}\end{aligned} \tag{7.1.31}$$

Again, these components have the same form as those of rotational distortion.

Thus we can expect to be able to solve a wide variety of distortion problems by considering the single case of an axis-symmetric perturbing force of the form given in equations (7.1.11), (7.1.25), and (7.1.31). We now consider some aspects of the solution of such problems.

## 7.2 Solution of Structure Equations for a Perturbing Force

The equations given by equations (7.1.1), and which arise from the perturbations discussed in Section 7.1, are partial differential equations and must be solved numerically. The numerical solution of partial differential equations constitutes a major area of study in its own right and is beyond the scope of this book. So we leave the numerical methods required for the actual solution to others and another time. Instead, we concentrate on the conditions required for the equations to have a solution and some of the implications of those solutions.

Since the perturbing forces derived in Section 7.1 are all conservative forces (that is, $\nabla \times \vec{D} = 0$), they are all derivable from some scalar potential which we can call $\Lambda$. This can be added to the gravitational potential so that we have a generalized potential to enter into the structure equations which we can call

$$\Phi(r, \theta, \phi) = \Omega + \Lambda \tag{7.2.1}$$

Since all the forces exhibited axial symmetry, there will be no explicit dependence of the generalized potential on $\phi$. There will be sets of values of $\theta$ and r, for which $\mathbf{\Phi}$ is constant. For the unperturbed gravitational potential alone these would be spheres of a given radius. For the generalized potential, they will be surfaces that exhibit axial symmetry. Such surfaces are known as *level surfaces* since a particle placed on one would feel no forces that would move it along the surface. Thus, if $\hat{n}$ represents a normal to such a surface, the gradient of the potential can be expressed as

$$\nabla\Phi = \hat{n}\frac{d\Phi}{dn} \tag{7.2.2}$$

As long as the chemical composition is constant, the state variables will be constant on level surfaces. This is sometimes known as *Poincare's theorem* which we prove for rotation in the next section. However, the result is entirely reasonable. The values of the state variables change in response to forces acting on the gas. Since the potential is constant on a level surface and its gradient is always normal to the surface, there are no forces along the surface to produce such differences.

If we take the state variables to be constant along level surfaces of constant potential, we can expect the variables to have the same functional dependence on the coordinates as the potential itself. Thus, from the form of $\Lambda$ given by equation (7.1.11), the state variables, and those parameters that depend directly on them, can be written as

$$
\begin{aligned}
\mathrm{P}(r,\theta) &= \mathrm{P}_0(r) + \mathrm{P}_2(r)P_2(\mathrm{Cos}\,\theta)\\
\rho(r,\theta) &= \rho_0(r) + \rho_2(r)P_2(\mathrm{Cos}\,\theta)\\
\epsilon(r,\theta) &= \epsilon_0(r) + \epsilon_2(r)P_2(\mathrm{Cos}\,\theta)\\
\kappa(r,\theta) &= \kappa_0(r) + \kappa_2(r)P_2(\mathrm{Cos}\,\theta)\\
\Omega(r,\theta) &= \Omega_0(r) + \Omega_2(r)P_2(\mathrm{Cos}\,\theta)
\end{aligned}
\tag{7.2.3}
$$

The gravitational potential must also be written with a $\theta$ dependence, because the perturbing force will rearrange the matter density so that the potential is no longer spherically symmetric.

We now regard equations (7.2.3) as perturbative equations in the traditional sense in that the terms with subscript 2 will be considered to be small compared to the terms with subscript $_0$.

### a Perturbed Equation of Hydrostatic Equilibrium

Substituting the perturbed form of the structure variables given by equation (7.2.3), into the equation of hydrostatic equilibrium [equation (7.1.1 d)], we get

$$
\begin{aligned}
\nabla P &= \nabla[\mathrm{P}_0(r) + \mathrm{P}_2(r)P_2(\mathrm{Cos}\,\theta)] = -\rho\,\nabla\Phi = -\rho\,\nabla\Omega + \rho\vec{D}\\
&= -\{\rho_0(r)\,\nabla\Omega_0(r) + \rho_0(r)\nabla[\Omega_2(r)P_2(\mathrm{Cos}\,\theta)]\\
&\quad + \rho_2(r)P_2(\mathrm{Cos}\,\theta)\,\nabla\Omega_0(r)\} + \rho_0(r)\vec{D}\\
&\quad + \rho_2(r)P_2^2(\mathrm{Cos}\,\theta)\,\nabla\Omega_2(r) + \rho_2(r)P_2(\mathrm{Cos}\,\theta)\vec{D}
\end{aligned}
\tag{7.2.4}
$$

The terms on the last line of equation (7.2.4) are small "second-order" terms by comparison to the other terms, so, in the tradition of perturbative analysis, we will ignore them. Since the equations must hold for all values of $\theta$, the r component of the gradient yields two distinct equations and the $\theta$ component yields one equation. These are basically the zeroth and second-order equations from the two components of the gradient. However, in general, there will be no zeroth-order $\theta$ equation, since the unperturbed state is spherically symmetric. Remembering the form for $\vec{\mathrm{D}}$ from equation (7.1.11), we see that the partial differential equations for hydrostatic equilibrium are

$$\frac{\partial P_0(r)}{\partial r} = -\rho_0 \frac{\partial \Omega_0(r)}{\partial r} + \rho_0(r)A(r)$$
$$\frac{\partial P_2(r)}{\partial r} = -\rho_0(r)\frac{\partial \Omega_2(r)}{\partial r} + \rho_2(r)\frac{\partial \Omega_0(r)}{\partial r} + \rho_0(r)B(r) \qquad (7.2.5)$$
$$P_2(r) = -\rho_0(r)\Omega_2(r) + \rho_0(r)C(r)/r$$

### b Number of Perturbative Equations versus Number of Unknowns

The number of independent partial differential equations generated by the vector equation of hydrostatic equilibrium is 3. In general, the vector equations of stellar structure will yield three such independent equations while the scalar equations will produce only two, since there is no $\theta$ component. In Table 7.1 we summarize the number of equations we can expect from each of the structure equations.

Each of the perturbed variables P, T, and Ω will produce a first- and second-order unknown function of r for a total of six unknowns. The perturbations in the density $\rho$ are not linearly independent since they are related to those of P and T by the equation of state. A similar situation exists for the opacity κ and energy generation ε. However the radiative flux is a vector quantity and will yield two unknown perturbed quantities, $F_{0r}$ and $F_{2r}$, from the r-component and one, $F_{1\theta}$, from the $\theta$ component. Thus the total number of unknowns in the problem is 9 and the problem is over determined and has no solution. This implies that we have left some physics out of the problem.

In counting the unknowns resulting from perturbing equations (7.1.1), we implicitly assumed that there were no mass motions present in the star, with the result that $\partial \boldsymbol{S}/\partial t$ in equation (7.1.1b) was taken to be zero. If we assume that a stationary state exists, then we can represent the local time rate of change of entropy by a velocity times an entropy gradient, so equation (7.1.1b) becomes

$$\nabla \cdot \vec{F} = \rho\epsilon - \vec{v} \cdot T\nabla S \qquad (7.2.6)$$

Thus, we have added a velocity with three components each of which will have two perturbed parameters. However, in general $\vec{v}_0$ will be zero, since we assume no circulation currents in the unperturbed model. In addition, $\nabla\boldsymbol{S}$ will exhibit axial symmetry and have no $\phi$ component. Thus the $v_{2\phi}$ perturbed parameter will be orthogonal to $\nabla\boldsymbol{S}$ and not appear in the final equations. This leaves us with 11 unknowns and 10 equations. However, we have not included the fact that mass conservation must be involved with any transport of matter, and modifying the conservation of mass equation to include mass motions will provide one more equation, completing the specification of the problem.

**Table 7.1 The Number of Independent Scalar Structure Equations**

| Equation | Zeroth-order $r$ | Second-order $r$ | Second-order $\theta$ | Total |
|---|---|---|---|---|
| (7.1.1*a*) | 1 | 1 | | 2 |
| (7.1.1*b*) | 1 | 1 | | 2 |
| (7.1.1*c*) | 1 | 1 | 1 | 3 |
| (7.1.1*d*) | 1 | 1 | 1 | 3 |
| Total | 4 | 4 | 2 | 10 |

## 7.3 Von Zeipel's Theorem and Eddington-Sweet Circulation Currents

For a solution to exist for the structure of a distorted star, we had to invoke mass motions in the star itself. This result was essentially obtained by von Zeipel[3] in the middle 1920s. At that time, the source of stellar energy was unknown, and von Zeipel set about to place constraints on the energy generation within a distorted star and in so doing produced one of the most misunderstood theorems of stellar astrophysics. The theorem is essentially a proof by contradiction that stars cannot simultaneously satisfy radiative and hydrostatic equilibrium if the stars are distorted. The normal version of the theorem is given for rigidly rotating stars and this is the version quoted by Eddington[4]. However, in the original publication, the version developed for rigid rotation is followed immediately by a version appropriate for tidally distorted stars[5]. Thus, clearly the theorem results from the induced distortion itself and is independent of the details that produce the distortion. We describe the version for rotation here, but keep in mind that is it the distortion that is important, not the mechanism by which that distortion is generated.

### a Von Zeipel's Theorem

As originally stated by von Zeipel[3] in 1924, this theorem says that for a rigidly rotating star in hydrostatic and radiative equilibrium, the rate of energy generation is given by

$$\epsilon = (\text{const})\left(1 - \frac{\omega^2}{2\pi G\rho}\right) \tag{7.3.1}$$

In light of what we now know about stars, this is an absurd result, because it requires that the energy generation rate become negative near the surface as the density goes to zero. As is the case when any theorem yields an absurd result, one must challenge the assumptions. To see where the trouble is likely to be, let us sketch von Zeipel's argument.

The equation of hydrostatic equilibrium

$$\nabla P = \rho\,\nabla\Phi \tag{7.3.2}$$

indicates that the potential gradient is related to the pressure gradient by the scalar density $\rho$. Hence, both vectors point in the same direction, and we can describe the change in pressure as a proportional change in potential so that

$$dP = \rho\, d\Phi \tag{7.3.3}$$

From this it is clear, that the pressure must be constant on a level surface where $\mathbf{d\Phi} = 0$. This is equivalent to saying that the pressure can be written as a function of the potential $\mathbf{\Phi}$ alone. If the pressure is a function of $\mathbf{\Phi}$ alone, then the scalar $\rho$, relating the potential and pressure gradients, must also be a function of $\mathbf{\Phi}$ alone. Or

$$\rho = \frac{dP(\Phi)}{d\Phi} = \rho(\Phi) \tag{7.3.4}$$

As long as the chemical composition $\mu$ is constant or at least not varying over an equipotential (level) surface, the ideal-gas law guarantees that the temperature will also be a function of $\mathbf{\Phi}$ alone:

$$T = \frac{P(\Phi)\mu m_h}{k\rho(\Phi)} = T(\Phi) \tag{7.3.5}$$

This is what we stated in Section 7.2 to be Poincare's theorem.

Now the radiative temperature gradient which arises from radiative equilibrium requires that

$$\vec{F} = \frac{4acT^3}{3\bar{\kappa}\rho}\,\hat{n} = -\left(\frac{c}{\bar{\kappa}\rho}\right)\nabla\left(\frac{aT^4}{3}\right) = -\frac{c}{\bar{\kappa}\rho}\frac{dP_r}{dn}\,\hat{n} \tag{7.3.6}$$

which, expressed in terms of the potential gradient, becomes

$$\vec{F} = -\frac{c}{\bar{\kappa}\rho}\frac{dP_r}{d\Phi}\frac{d\Phi}{dn}\,\hat{n} = -\frac{c}{\bar{\kappa}\rho}\frac{dP_r}{d\Phi}\nabla\Phi \tag{7.3.7}$$

However, since $\bar{\kappa}$, $\rho$, and T are all state variables or functions of them, they are all functions of $\mathbf{\Phi}$ alone and

$$\vec{F} = f(\Phi)\,\nabla\Phi \tag{7.3.8}$$

But $\nabla\mathbf{\Phi}$ is just the local gravity, and it is most certainly not a function of the potential alone or constant on level surfaces. Indeed, for a critically rotating star, the gravity varies from the mass gravity at the pole to zero at the equator, where the mass gravity is balanced by the centripetal acceleration. Thus, equation (7.3.8) basically says that in the presence of the radiative temperature gradient

$$|\vec{F}| = (\text{const})|\vec{g}| \tag{7.3.9}$$

which is sometimes known as *von Zeipel's law of gravity darkening*.

If we further consider radiative equilibrium in the absence of mass motions, we can write

$$\nabla \cdot \vec{F} = \rho\epsilon = \nabla \cdot [f(\Phi)\,\nabla\Phi] = f(\Phi)\,\nabla^2\Phi + \nabla f(\Phi) \cdot \nabla\Phi \tag{7.3.10}$$

For a star in rigid rotation, $\nabla^2\mathbf{\Phi}$ will depend on only the density and some constants and so will be a function of $\mathbf{\Phi}$ alone. The left-hand side of equation (7.3.10) will depend on only the state variables and must also be a function of $\mathbf{\Phi}$ alone. But, again, the gravity $\nabla\mathbf{\Phi}$ is not a function of $\mathbf{\Phi}$ alone, so

$$\nabla f(\Phi) = 0 \tag{7.3.11}$$

Therefore, evaluating $\nabla^2\mathbf{\Phi}$ by means of equations (7.1.10) and (7.2.1), we get

$$\epsilon = (\text{const})\frac{\nabla^2\Phi}{\rho} = (\text{const})\left(1 - \frac{\omega^2}{2\pi G\rho}\right) \tag{7.3.12}$$

The absurdity of equation (7.3.12) results primarily from the assumption that the effects of mass motions are not present in equation (7.3.10). The addition of mass motions removes the exclusive dependence of radiative equilibrium on the potential and the remainder of the argument falls apart releasing the constraint on $\varepsilon$. The gradient of f($\mathbf{\Phi}$) is no longer zero and allows for the variation of $\varepsilon$ with radius that we know must exist. However, as we shall see, small amounts of energy are all that is required to be carried by the currents of the mass motions. Thus the radiative gradient will still be basically the temperature gradient that is operative in the radiative zones of the star. The result given in equation (7.3.9) will still be largely correct, and we may expect the radiative flux to be redistributed in accordance with the local value of the gravity. Therefore, particularly for the rapidly rotating upper main sequence stars with radiative envelopes, we may expect that their surface will not be uniformly bright, but will become darker with decreasing local gravity. While it is true that the conditions of radiative equilibrium become rather different in the stellar atmosphere as the photons begin to escape into outer space, the thickness of the atmosphere compared to the depth of the radiative envelope is minuscule and whatever variation of radiative flux has been established at the base of the atmosphere will be largely reflected in the flux emerging from the star. So von Zeipel's theorem, while telling us nothing about the energy generation within the star, does tell us quite a lot about the manner in which the radiation leaves the star.

### b Eddington-Sweet Circulation Currents

We have seen that radiative and hydrostatic equilibrium cannot be simultaneously satisfied in a distorted star and that the failure of these conditions results in the mass motion of material carrying energy to make up the deficit produced by the departure from spherical geometry. That the energy transfer is accomplished by means of the physical motion of material seems ensured. There simply is no other mechanism to effect the transfer. Radiation has been accounted for, conduction is ineffective and the environment is stable against classical convection. These arguments persuaded Eddington[4] (p. 286) to suggest the existence of such currents which were later quantified by Sweet[6]. Let us now estimate the speed of these currents and determine the amount of energy they may carry. The currents will be quite slow since, even in the most distorted of stars the local departure of the energy flux from spherical symmetry is quite small. Even so, any mass motion could be important if it transports material throughout the star on a nuclear time scale. The possibility would then exist for a resupply of nuclear fuel, and that could upset some of our stellar evolution calculations.

***Conservation of Energy and Circulation Velocity*** The distortion of a star will force a departure from radiative equilibrium and a change in the divergence of the radiative flux from that expected for spherical stars. We argued earlier that the change in the divergence will be brought about by the additional nonradial transport of energy by mass motions, as expressed by the second term on the right-hand side of equation (7.2.6). Thus, to estimate the velocity of those motions, we must estimate the entropy gradient that distortion will establish.

From thermodynamics remember that the entropy can be expressed in terms of the state variables of an ideal gas as

$$S = C_p \ln T - n\mathrm{R} \ln P + S_0 \qquad (7.3.13)$$

Therefore, the general energy source term in equations (7.1.1b) can be written as

$$T\frac{\partial S}{\partial t} = C_P\frac{\partial T}{\partial t} - \frac{\partial P}{\partial t} \tag{7.3.14}$$

and the entropy gradient of equation (7.2.6) becomes

$$T\nabla S = C_P\nabla T - \nabla P \tag{7.3.15}$$

Now the temperature and pressure gradients are both normal to equipotential surfaces, so the vector nature of equation (7.3.15) is unimportant and it must hold for the magnitude of the individual terms. Therefore,

$$T|\nabla S| = C_P\rho|\nabla P|\left(\frac{1}{\rho}\frac{|\nabla T|}{|\nabla P|} - \frac{1}{\rho C_P}\right) \tag{7.3.16}$$

Equation (7.3.16), when combined with the ideal-gas law and the fact that the temperature and pressure gradients point in the same direction, enables equation (7.3.15) to be written as

$$T|\nabla S| = C_P\rho\left(\frac{T}{P}\right)|\nabla P|\left(\frac{P}{\rho T}\frac{dT}{dP} - \frac{P}{C_P\rho T}\right) \tag{7.3.17}$$

Now the adiabatic gradient can be expressed as

$$\left.\frac{dT}{dP}\right|_{\rm ad} = \frac{T}{P(n+1)} = \frac{P/(kT)}{\rho C_P} \tag{7.3.18}$$

Since for the zeroth-order values of these gradients

$$\left(\frac{dT}{dP}\right)_0 = \frac{(dT/dr)_0}{(dP/dr)_0} \tag{7.3.19}$$

we can write the zeroth-order value for the entropy gradient as

$$T|\nabla S|_0 = \left[|\nabla P|\left(\frac{dT}{dP}\right)_{\rm ad}^{-1}\left(\frac{dT}{dP} - \left.\frac{dT}{dP}\right|_{\rm ad}\right)\right]_0 = \left(|\nabla P|\frac{\Delta\nabla T}{\nabla T|_{\rm ad}}\right)_0 \tag{7.3.20}$$

In the equilibrium model, there are no mass motions; the velocity in equation (7.2.6) is already a first-order term and so to estimate its value we need only keep zeroth-order terms in the entropy gradient. The zeroth-order pressure gradient is just

$$T|\nabla S|_0 = \left[|\nabla P|\left(\frac{dT}{dP}\right)_{\rm ad}^{-1}\left(\frac{dT}{dP} - \left.\frac{dT}{dP}\right|_{\rm ad}\right)\right]_0 = \left(|\nabla P|\frac{\Delta\nabla T}{\nabla T|_{\rm ad}}\right)_0 \tag{7.3.21}$$

Combining this with equations (7.2.6) and (7.3.20), we can write the perturbed equation for energy conservation as

$$\nabla \cdot \vec{F} = \rho\epsilon - \vec{g}_0 \cdot \vec{v}_2 \rho_0(r) \frac{(\Delta \nabla T)_0}{(\nabla T|_{ad})_0}$$
$$= \rho_0\epsilon_0 - g_0 v_{2,0}(r)\rho_0(r) \frac{(\Delta \nabla T)_0}{(\nabla T|_{ad})_0} \tag{7.3.22}$$

Now, from von Zeipel's gravity darkening law [equations (7.3.6) and (7.3.7)] we have

$$\vec{F} = -\left(\frac{4acT^3}{3\bar{\kappa}\rho}\frac{dT}{d\Phi}\right)\vec{g} \tag{7.3.23}$$

which means that we can write the divergence of the flux as

$$\nabla \cdot \vec{F} = \nabla\Phi \cdot \frac{d}{d\Phi}\left(-\frac{4acT^3}{3\bar{\kappa}\rho}\frac{dT}{d\Phi}\nabla\Phi\right)$$
$$= -\nabla\Phi \cdot \nabla\Phi\left[\frac{d}{d\Phi}\left(\frac{4acT^3}{3\bar{\kappa}\rho}\frac{dT}{d\Phi}\right)\right] - \frac{4acT^3}{3\bar{\kappa}\rho}\frac{dT}{d\Phi}\nabla \cdot \nabla\Phi \tag{7.3.24}$$

But, since the radiative flux and gravity are vectors pointing in the same direction,

$$\nabla \cdot \vec{F} = |\vec{g}|^2\left[\frac{d(F/g)}{d\Phi} + \frac{F}{g} \cdot \nabla^2\Phi\right] \tag{7.3.25}$$

For rotation we can obtain the generalized potential from equations (7.1.1a) and (7.1.10). Expressing the rotational potential in cylindrical coordinates, we get

$$\nabla^2\Phi = 4\pi G\rho - s^{-1}\frac{\partial}{\partial s}\left[s\frac{\partial}{\partial s}\left(\frac{1}{2}\omega^2 s^2\right)\right] = 4\pi G\rho - s^{-1}\frac{\partial}{\partial s}(\omega^2 s^2) \tag{7.3.26}$$

Equation (7.3.24) for the perturbed flux divergence can be broken into its perturbed components so that

$$\nabla \cdot \vec{F}_0 = \frac{d(F_0/g_0)}{d\Phi} g_0^2 + \frac{F_0}{g_0}\left\{4\pi G\rho - \left[\frac{1}{s}\frac{d}{ds}(\omega^2 s^2)\right]_0\right\}$$
$$\nabla \cdot \vec{F}_2 = \frac{d(F_0/g_0)}{d\Phi} 2g_0 g_2 - \frac{F_0}{g_0}\left[\frac{1}{s}\frac{d}{ds}(\omega^2 s^2)\right]_2 \tag{7.3.27}$$

Since the zeroth-order flux-to-gravity ratio can be written as

$$\frac{F_0}{g_0} = \frac{L(r)/(4\pi r^2)}{GM(r)/r^2} = \frac{L(r)}{4\pi GM(r)} \tag{7.3.28}$$

its derivative with respect to the generalized potential is

$$\frac{d(F_0/g_0)}{d\Phi} = -\frac{1}{g_0^2}\frac{\rho_0 L(r)}{M(r)} \tag{7.3.29}$$

The luminosity can be written in terms of an average energy generation ratee and *M*(r) so that

$$L(r) = \int_0^r 4\pi r^2 \rho\epsilon\, dr \equiv \bar{\epsilon}M(r) \tag{7.3.30}$$

which yields

$$\nabla \cdot \vec{F}_2 = -\bar{\epsilon}\rho_0 \frac{2g_2}{g_0} - \frac{L(r)}{4\pi GM(r)}\left[\frac{1}{s}\frac{d}{ds}(\omega^2 s^2)\right]_2 \tag{7.3.31}$$

If we further assume that the distortion is small so that $g_2/g_0 << 1$, then equation (7.3.31) can be combined with equation (7.3.22) to give the velocity for the induced circulation currents as

$$v_c \equiv v_{2,0}(r) = \frac{(\nabla T|_{\mathrm{ad}})_0}{(\Delta\nabla T)_0}\frac{L(r)}{4\pi g_0 \rho GM(r)}\left[s^{-1}\frac{d(\omega^2 s^2)}{ds}\right]_2 \tag{7.3.32}$$

***Eddington-Sweet Time Scale and Mixing*** If we take reasonable values for the parameters in equation (7.3.32), namely,

$$g_0 = \frac{GM}{R^2} \qquad L(r) = L \qquad \omega = \text{const} = w\left(\frac{8GM}{27R^3}\right)^{1/2} \tag{7.3.33}$$

then we can rewrite the circulation velocity as

$$v_c = \frac{(\nabla T|_{\mathrm{ad}})_0}{(\Delta\nabla T)_0}\frac{\bar{\rho}}{\rho}\frac{16R^2 L}{81GM^2}w^2 \tag{7.3.34}$$

Here we have introduced the fractional angular rotational velocity *w*, which is just ω normalized by the critical angular velocity, $\omega_c$, where the effective equatorial gravity is zero for a centrally condensed star (Roche model). In addition, if we introduce a time scale such as the Kelvin-Helmholtz time scale [equation (3.2.11)], we can further reduce the expression for the circulation velocity to some dimensionless ratios multiplied by $R/\tau_{K\text{-}H}$ and get

$$v_c \approx \frac{(\nabla T|_{ad})_0}{(\Delta\nabla T)_0} \frac{\bar{\rho}}{\rho} \frac{16w^2}{135} \frac{R}{\tau_{K\text{-}H}} \tag{7.3.35}$$

If we introduce the Eddington-Sweet time scale as the time required for the circulation currents to carry the material a distance R, then

$$\tau_{E\text{-}S} = \tau_{K\text{-}H} \frac{135}{16w^2} \frac{\rho}{\bar{\rho}} \frac{(\Delta\nabla T)_0}{(\nabla T|_{ad})_0} \tag{7.3.36}$$

The sun is a rather slowly rotating star, and if the angular velocity of the core is displayed on the surface, there is little rotational distortion, so the Eddington-Sweet currents should be small. Certainly the core density will exceed the mean density, and we have already indicated that the radiative core of the sun is barely stable against convection. Thus, the following values for the solar radiative core should provide fair estimates of the internal conditions necessary for the evaluation of the circulation currents:

$$w_\odot^2 \sim 10^{-5} \qquad \frac{\rho(\text{core})}{\bar{\rho}} > 1 \qquad \frac{(\Delta\nabla T)_0}{(\nabla T|_{ad})_0} \sim 1 \tag{7.3.37}$$

These values, when substituted into equation (7.3.35), yield

$$\tau_{E\text{-}S} \geq 10^5 \tau_{K\text{-}H} > \tau_n \tag{7.3.38}$$

So the material in the core takes much longer than the nuclear time scale to circulate, and we would not expect the core of the sun, or solar-type stars that are slowly rotating to be mixed. So the stellar evolution scenarios we developed for lower main sequence stars in Chapter 5 remain intact.

The situation is less clear for upper main sequence stars. Here the envelope has a density is lower than the mean density, is in radiative equilibrium, and in danger of supplying fuel to the core. In addition, many of these stars rotate very rapidly so that we might expect much larger circulation current velocities. Reasonable values of the parameters in equation (7.3.35) for rapidly rotating B stars are

$$w_B^2 \approx 1 \qquad \frac{\rho(\text{env})}{\bar{\rho}} < 1 \qquad \frac{(\Delta\nabla T)_0}{(\nabla T|_{ad})_0} \sim 1 \tag{7.3.39}$$

which lead to

$$\tau_{E\text{-}S}(B\ \text{star}) \sim \tau_{K\text{-}H} < \tau_n \tag{7.3.40}$$

On the basis of this analysis we would have to conclude that there is an excellent possibility that rapidly rotating stars on the upper main sequence may be mixed thoroughly throughout and their main sequence life times may be prolonged. However, we have not dealt with the formation of the helium core itself and the effects caused by the change in chemical composition.

## 7.4 Rotational Stability and Mixing

A complete discussion of the stability of a rotating star is quite complicated and beyond the scope of this book. However, we consider some of the important effects on the stability of rotating stars. The usual approach to the subject of stability involves finding the spectrum of perturbations for which the equations of motion are stable (i.e., the perturbations will damp out with time). A related approach is to use the Virial theorem[7], which after all is just a spatial moment of the equations of motion. Various physical processes may occur and give rise to an instability:

1. Buoyancy forces that result from thermal stratification
2. Perturbations that may grow in the presence of an angular momentum gradient
3. Instabilities in the presence of a magnetic field
4. Shear instabilities producing flows both parallel and perpendicular to the local gravity field
5. Failure of the equipotential surfaces being surfaces of constant temperature and pressure [that is, $\omega \neq \omega(s)$]
6. Development of a molecular weight gradient as a result of nuclear evolution
7 Diffusion of heat, angular momentum, and the mean molecular weight

Of all these effects, probably the most important for the stability of rotating stars is the various shear instabilities.

### a Shear Instabilities

The existence of a velocity gradient implies the presence of particle interactions resulting from changes in the macroscopic velocity field. These interactions result from the collisions that are the product of the differential stream motion of the gas, and the severity of these collisions is usually characterized by the viscosity n of the material. The viscosity will try to remove the velocity gradient. However, if the shear is too great, the velocity field will break up into turbulent flow. The conditions of the flow can be characterized by a dimensionless number known as the *Reynolds number* $R_e$, which for rotating stars is

$$R_e \approx \frac{\omega R_*^2}{\nu} \tag{7.4.1}$$

Should this number exceed a critical value, known as the *critical Reynolds number*, which is about $10^3$, the flow will break up into turbulent eddies and the smooth macroscopic motion will become chaotic.

It is useful to break the notion of shear motion into two limiting cases. Motion along the equipotential surfaces will be unopposed by gravity and any of the phenomena that arise from the gravity field. Thus, perturbations that produce horizontal shear can grow unopposed except by the dissipative forces that arise from the viscosity of the gas. However, shear instabilities that arise from motions perpendicular to the equipotential surfaces must overcome forces caused by the temperature and perhaps molecular weight gradients. Thus, the star will be much more stable against vertical shear instabilities, and the time scales for their respective growths will be quite different. For the horizontal shear instabilities the time scale is dominated by the viscosity, while for vertical shear instabilities the time scale for development is essentially the thermal, or Kelvin-Helmholtz, time scale. Thus,

$$t_h \approx \frac{R_*^2}{\nu} \qquad t_v \approx t_{\text{K-H}} \tag{7.4.2}$$

The nature of the viscosity of stellar material has long been a subject of heated debate. If one calculates the viscosity simply on the basis of the collisional interaction of the atoms of the gas, one will obtain an extremely small number and an associated growth time scale which is long compared to the nuclear time scale for the star. However, if the flow becomes turbulent, then the dominant collisions occur, not between atoms, but between turbulent elements, giving rise to a "turbulent viscosity" which is many orders of magnitude greater than the kinematic viscosity of the atoms themselves. Unfortunately, the theory of turbulent flow is not sufficiently developed to yield reliable values for the turbulent viscosity, so we must rely on empirical values for systems with dimensions vastly smaller than those of stars. Nevertheless, the prevailing opinion seems to be that turbulent viscosity will be many orders of magnitude greater than kinematic viscosity and so shear instabilities will be of considerable importance in bringing about the redistribution of angular momentum within the star.

From the arguments in Section 7.3 [equation (7.3.40)] it seemed likely that the Eddington-Sweet circulation currents could redistribute material and angular momentum on a time scale comparable to the Kelvin-Helmholtz time for a rapidly rotating star. This is the same order of magnitude as the time scale for the development of the vertical shear instabilities. However, it is rather greater than the time scale for the horizontal shear instabilities should they result from turbulent flow.

So these horizontal shear instabilities would appear to be the dominant phenomenon that redistributes the material and angular momentum in the most rapidly rotating stars. This would lead to a steady-state rotation law where the angular velocity was constant on equipotential surfaces and had a condition for stability of the form

$$\frac{\partial^2(\omega s^2)^2}{\partial(\mathrm{Cos}\,\theta)^2} \neq 0 \tag{7.4.3}$$

The most plausible rotation law that would satisfy these constraints is rigid rotation, and this may well be the only equilibrium law for rapidly rotating stars. However, many questions must to be answered before it can be determined if this law actually exists in these stars.

### b Chemical Composition Gradient and Suppression of Mixing

composition m did not appear on the right-hand side of equation (7.3.5). In an evolving star, the chemical composition is continually changing as a result of nuclear processes. Thus, for the early-type stars, we expect the convective core to change its chemical composition on a nuclear time scale, causing m to increase with time. This will lead to a discontinuity in the chemical composition at the core-envelope interface. Now imagine a blob of helium displaced upward by the circulation currents into the less dense hydrogen envelope. The forces of hydrostatic equilibrium will tend to restore the higher-density helium to the core, while the circulation currents will try to mix the helium higher in the hydrogen envelope. Fricke and Kippenhahn[8] have shown that ratio of the circulation velocity to the restoring velocity induced by hydrostatic equilibrium is given by

$$\frac{v_c}{v_\mu} \sim \frac{0.3w^2}{\Delta\mu/\mu_c} \tag{7.4.4}$$

Since the greatest value of $w$ which is allowed is unity, and since a pure helium core will produce $\Delta\mu / \mu_c$ . 0.5, we would expect the core-envelope interface to be stable against any vertical motion that would allow mixing. For the typical B star where $w \approx 0.4$, a reasonably small gradient in the chemical composition will stabilize the star against rotationally driven mixing, so we may expect the stellar evolution scenarios for the upper main sequence stars described in chapter 5 to remain correct. *(Recently some two- and three- dimensional model interior calculations have cast doubt on this conclusion, but the issue is far from definitively resolved).*

### c Additional Types of Instabilities

Conditions that can lead to instability in a rotating star seem so numerous that some physicists have despaired from finding any angular momentum distribution that is stable for the lifetime of the star, and it may well be true that no such distribution exists. The number and type of instabilities that can occur are indeed legion. However, what is relevant for the theory of stellar evolution is the time scale for the development of these instabilities and what they do to the star. For rapidly rotating stars, shear instabilities are likely to occur and lead to a rotation law where the angular velocity is constant on equipotential surfaces. There are additional constraints on the rotation law. Should an outward displacement that conserves angular momentum produce a perturbation that has a greater angular velocity than the local velocity field, the perturbation will be dynamically unstable and will grow on the dynamical time scale. This basically geometric instability is sometimes called the *Solberg-Hφiland instability*, and it constrains the angular momentum per unit mass so that

$$\left.\frac{\partial(\omega s^2)^2}{\partial s}\right|_{\rho=\text{const}} > 0 \tag{7.4.5}$$

Thus, angular velocity laws that decrease faster than $s^{-2}$ will be dynamically unstable. A similar criterion holds for the *Goldreich-Schubert-Fricke instability*. However, the time scale for its development is very much longer because this instability basically arises from the removal of buoyancy stabilization of the temperature gradient by thermal diffusion. If we add to the angular velocity constraints the notion that the rotation law should be derivable from a potential [equation (7.1.13)], then the constraints on the angular velocity distribution become

$$\frac{\partial(\omega s^2)^2}{\partial s} > 0 \qquad \frac{\partial(\omega s^2)^2}{\partial z} = 0 \tag{7.4.6}$$

The notion that the rotation law should be conservative is largely based on personal prejudice and will be wrong if dissipative forces like those arising from viscosity are present. Under these conditions the criterion for stability becomes

$$\left\{\frac{\partial[\ln(\omega s^2)^2]}{\partial z}\right\}^2 \geq \frac{\nu}{kT}\frac{g_{\text{eff}}}{\omega^2(s)} \tag{7.4.7}$$

Since the quantity $\nu/kT$ is usually quite small for stars [i.e., of the order of $10^{-6}$(cgs)], the Goldreich-Schubert-Fricke instability is unimportant except in cases of slow rotation and long nuclear time scales. Thus, this instability has been applied to the sun with some interesting results. However, it can be easily stabilized by a molecular weight gradient such as that described by equation (7.4.4).

Under conditions of rapid rotation, one might expect non-axis symmetric motions to occur that can separate surfaces of constant pressure from equipotential surfaces. Instabilities resulting from such situations are generally referred to as *baroclinic instabilities*. These and other types of diffusive instabilities we leave to others to discuss.

## Problems

1. Discuss the problems you would encounter in describing the structure of a very rapidly rotating magnetic neutron star. Specifically discuss how you would propose calculating a model of the structure, and list the assumptions you would make.

2. Show that

$$\frac{v_c}{v_\mu} \sim \frac{0.3w^2}{\Delta\mu/\mu_c}$$

and clearly state the assumptions you would make.

3. Indicate how the conservation of mass equation should be modified to accommodate the flow of matter resulting from the Eddington-Sweet currents.

## References and Supplemental Reading

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8. Fricke, K.J., and Kippenhahn, R.: *Evolution of Rotating Stars*, Annual Review of Astronomy and Astrophysics, Annual Review, Palo AltoCalif. 1972, Ed: L. Goldberg Vol. 10, pp45-72.

During the last quarter of a century, much has been done regarding the structure of distorted stars. A useful historical review through the early 1970's can be found in

Roxburgh, I.W.:"*Rotation and Stellar Interiors*" *Stellar Rotation*, Ed: A. Slettebak,D. Reidel Pub. Co.,Dordrecht-Holland, 1970, p9-19.

However, the most comprehensive review of the problems relating to the structure of rotating stars is

Toussel, J.L.: *The Theory of Rotating Stars* Princeton University Press, Princeton N.J., 1978.

For the fundamental literature on distorted polytropes, see

Chandrasekhar, S.: *The Equilibrium of Distorted Polytropes I (The Rotational Problem)*, Mon. Not R. ast. Soc. 93, 1933, pp.390-405.

Chandrasekhar, S.: *The Equilibrium of Distorted Polytropes II (The Tidal Problem),* Mon. Not. R. astr. Soc. 93, 1933, pp.449-471.

More recent work on this subject can be found in Limber and Roberts (1965) and Geroyannis and Valvi (1987) (see References and Supplemental Reading in Chapter 2).