

12

The Construction of a Model Stellar Atmosphere

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12.1 Statement of the Basic Problem

We have now acquired all the material necessary to construct a model of the atmosphere of a star. This material includes not only the dependence of the state variables P , T and ρ with depth in the atmosphere but also an approximation to the emergent spectrum. That predicted spectrum will not contain the details of the stellar absorption lines, but will show the departures from the Planck function of a radiation field in thermodynamic equilibrium. The major departures are caused by the absorption edges corresponding to the ionization limits for the elements included in the calculation. Even if there were no such discontinuities in the frequency dependence of the absorption coefficient, the emergent spectra would still differ from those of a blackbody. Since the photons emerge from different depths in the atmosphere, having different temperatures, even a gray atmosphere spectrum will depart from the Planck function. The more sophisticated spectrum results from the solution of the equation of radiative transfer, the calculation of which represents a major part of the construction of a model atmosphere.

In developing this material, we have used the same conservation laws that yielded the equations of structure for the stellar interior. However, the resulting formulation is somewhat different. The conservation of momentum yielded the expression for hydrostatic equilibrium, as it did in the stellar interior. However, the assumption of a plane-parallel structure for the atmosphere and the use of a different depth coordinate have caused the expression of hydrostatic equilibrium to take a somewhat different form than for the stellar interior. The conservation of energy is at the root of radiative equilibrium. This condition is imposed on the Boltzmann transport equation itself which was used to produce the equation of radiative transfer. However, because of the departure of the radiation field from STE, the flow of radiation was described by an integral equation, implying that the solution at any point depends on the solution at all points. As a result, we no longer have the simple differential equation for the radiative gradient that was appropriate for the stellar interior. Even the equation of state, which results from saying that the local velocity field of the particles is largely isotropic and dominates any macroscopic flow velocity, is present in basically the form used in the stellar interior. Although the calculation of the mass absorption coefficient appears to present a greater problem for stellar atmospheres, this is largely an illusion. The construction of an accurate model interior requires careful calculation of the frequency-dependent absorption coefficient, and the range of atomic phenomena that must be included is actually greater than that of an atmosphere because of the greater range of possible ionization states. However, in the stellar atmosphere, the frequency dependence of the absorption coefficient enters far more directly into the solution and plays a greater role. The use of the Rosseland Mean opacity for stellar interiors tends to average out the "mistakes" in the opacity calculations whereas those mistakes in a stellar atmosphere are directly visible in the emergent spectrum. The presence of molecules is an added complication for the theory of stellar atmospheres that does not plague the theory of stellar interiors.

Much has been concealed by writing the opacity as a function of the state variables. But while the details are messy and LTE has been assumed, the process is straightforward. The major difference between the calculation of a stellar interior and the construction of a model stellar atmosphere can be seen in the last of equations (12.1.1). No longer do we have a situation that can be mathematically described as a linear two-point boundary value problem. Because of the assumption of plane parallelism the "eigenvalues" of the problem have been reduced to two, T_e and g . In addition, the four nonlinear differential equations of the interior structure have been replaced by one first order differential equation and an integral equation for the source function from which all physically relevant moments of the radiation field can be calculated. The global nature of this integral equation forces a rather different approach to the construction of a model stellar atmosphere from that adopted for the stellar interior.

We can summarize the equations of atmospheric structure, obtained from these conservation laws and assumptions:

$$\begin{aligned}
 \frac{dP}{d\tau_v} &= \frac{g}{\kappa_v + \sigma_v} && \text{hydrostatic equilibrium} \\
 d\tau_v &= -(\kappa_v + \sigma_v)\rho dx && \text{definition of } \tau_v \\
 P &= \frac{\rho kT}{\mu m_h} + P_r && \text{equation of state} \\
 F &= \frac{\sigma T_e^4}{\pi} && \text{definition of } T_e \\
 \int_0^\infty \kappa_\nu \{J_\nu(\tau_\nu) - B_\nu[T(\tau_\nu)]\} d\nu &= 0 && \text{radiative equilibrium} \\
 S_\nu(\tau_\nu) &= \epsilon_\nu(\tau_\nu)B_\nu[T(\tau_\nu)] + \frac{1}{2}[1 - \epsilon_\nu(\tau_\nu)] \int_0^\infty S_\nu(t)E_1|\tau_\nu - t| dt && \text{integral equation of radiative transfer}
 \end{aligned}
 \tag{12.1.1}$$

In general, we split the problem into two parts, each of which can be solved with the knowledge of the other. After making as shrewd a guess as possible for the approximate solution of one, we solve the other and use those results to improve the initial guess for the first. We can then proceed to solve these two halves of the problem alternately until we obtain an answer that is self-consistent and satisfies the conditions of radiative equilibrium throughout the atmosphere. The basic division of the problem is to calculate the depth dependence of the state variables under the assumption of the radiation field and then to use this atmospheric structure to improve the radiation field. Since the initial guess for the radiation field is not likely to be correct, we cannot expect that radiative equilibrium will be satisfied throughout the atmosphere. Thus we try to use the calculated departures from radiative equilibrium to modify the physical structure so as to produce a radiation field that more nearly satisfies radiative equilibrium. Since we have already dealt with the solution of the equation of radiative transfer, most of this chapter involves the iterative aspect of the problem. Proper formulation of such a correction scheme will provide the basis for forming a rigorous iterative algorithm that will converge to a fully self-consistent model atmosphere with a structure that yields a radiation field satisfying radiative equilibrium throughout the entire atmosphere. However, we must begin with some comments on how to find the dependence of state variables on depth in the atmosphere, given the radiation field. This involves the solution of the differential equation for hydrostatic equilibrium.

12.2 Structure of the Atmosphere, Given the Radiation Field

At the outset of any atmosphere calculation we must decide on the particular atmosphere to be modeled. Choosing the parameters T_e , g , and μ is analogous to choosing M , L , R , and μ for the construction of a model stellar interior. Indeed, the relationship between them is straightforward:

$$T_e = \left(\frac{L}{4\pi\sigma R^2} \right)^{1/4} \quad g = \frac{GM}{R^2} \quad (12.2.1)$$

a Choice of the Independent Variable of Atmospheric Depth

Before beginning the model calculation itself, we must choose a depth parameter to serve as an independent variable. While the traditional choice has always been the optical depth, some more modern atmosphere codes use a "density depth" like

$$h(x_0) = \int_0^{x_0} \rho(x) dx \quad (12.2.2)$$

as the independent depth variable. As we see, this greatly simplifies the calculation of hydrostatic equilibrium, but introduces some difficulties in the solution of the equation of transfer. For the most part, we use the traditional optical depth as our independent depth variable.

Which optical depth should we use? Early investigators would pick one of the mean opacities to generate a mean optical depth, and this dimensionless, frequency-independent variable would provide an excellent parameter for describing the atmospheric structure. Unfortunately, the calculation of the mean opacity at numerous depths in the atmosphere is a nontrivial undertaking and is completely avoidable. Since there is no particular significance for any of the mean opacities, no optical depth scale is to be preferred over any other on the basis of the physical information contained there. Therefore, it makes good sense to pick a monochromatic optical depth at some frequency $\tau(\nu_0)$ (or simply τ_0) as the depth parameter, thereby avoiding the tedious calculation of the mean opacity and the associated mean optical depths. However, since the radiative transfer equation must be solved at each frequency, it will be necessary to interpolate the solution to the reference optical depth τ_0 . So it would be wise to choose a frequency in the general vicinity of the maximum energy flow through the atmosphere and in a part of the spectrum where the opacity does not vary rapidly with frequency. This will tend to minimize interpolation errors when the solutions are transferred to the reference optical depth. For the majority of the development in this chapter, this is the choice that we make. The relevant optical depths are then given by

$$\begin{aligned} d\tau_0 &= (\kappa_0 + \sigma_0) \rho dx \\ d\tau_\nu &= \frac{\kappa_\nu + \sigma_\nu}{\kappa_0 + \sigma_0} d\tau_0 \equiv k_\nu d\tau_0 \end{aligned} \quad (12.2.3)$$

The parameter κ_ν is just a normalized opacity which relates the differential monochromatic optical depth to its counterpart on the reference depth scale.

b Assumption of Temperature Dependence with Depth

Having specified the nature of the atmosphere and chosen the depth parameter, we can begin the calculation of the atmospheric structure with depth. We have indicated that we will split the process of making the model into two parts by assuming the results of the radiative transfer solution in order to calculate the atmospheric structure. The form that this assumption takes is the dependence of the temperature with depth. That is, the end result for the solution of the radiative transfer calculation will be to generate the dependence of the temperature with depth in a manner that is consistent with radiative equilibrium. Thus, to begin our calculation, we must assume the existence of this temperature distribution. We may obtain this information either as a result of an earlier model calculation or from an initial approximation.

In Section 10.2, we spent considerable effort in solving the equation of transfer for the gray atmosphere. One of the results of this effort was the temperature distribution in the Eddington approximation [equation (10.2.16)]. Remembering that for a gray atmosphere in radiative equilibrium

$$S(\tau) = J(\tau) = B(\tau) \tag{12.2.4}$$

we can write the more general result

$$T^4(\tau) = \frac{3}{4} T_e^4 [q(\tau) + \tau] \tag{12.2.5}$$

where $q(\tau)$ is the Hopf function specified in equation (10.2.34). Although the gray atmosphere does not specify a unique physical atmospheric structure, it does provide a temperature distribution that scales with the effective temperature and is consistent with radiative equilibrium. In addition, the opacity in a wide variety of stars is relatively independent of frequency over a large part of the spectrum, so that the gray atmosphere temperature distribution provides a good first approximation to the actual temperature distribution. However, the accuracy of this assumption does depend on the choice of reference optical depth being representative of the atmosphere as a whole, so we can only expect it to form an approximate first guess.

c Solution of the Equation of Hydrostatic Equilibrium

The equation of hydrostatic equilibrium is a deceptively simple looking first-order differential equation. There are many sophisticated methods for obtaining the numerical solution to such an equation, but all such methods involve knowing at least one value (and usually several values) for the solution at and near

the boundary. This poses both philosophical and practical problems. The boundary of a plane-parallel atmosphere, while well located in optical depth, is poorly placed in physical depth since it formally occurs where the density vanishes. In principle, this occurs at an infinite distance from the star where the plane-parallel approximation itself would no longer be valid. However, in practice, a small but finite optical depth is reasonably located with respect to the photosphere (i.e., near optical depth $\tau_0/3$) so that boundary conditions can be specified there without jeopardizing the accuracy of the solution. Since we have assumed a distribution of temperature with optical depth, there is no problem in determining the boundary temperature.

In Section 4.1a we discussed how to relate the mass fractions of hydrogen, helium, and "metals" to the mean molecular weight and thereby provide a connection between the number and mass density. The Saha equations for each element [equation (11.1.16)] provide a relationship between the relative ionization fraction, the temperature, and the electron pressure. By remembering that the sum of all the various states of ionization for a particular element is simply equal to the number abundance for the element, it is possible to parameterize the opacity in terms of the gas pressure, temperature, and electron pressure. Thus, we may write the total pressure at any optical depth as

$$P(\tau_i) = P_g[T(\tau_i), P_e(\tau_i)] + P_r[T(\tau_i)] \quad (12.2.6)$$

In a similar manner it is possible to integrate the equation of hydrostatic equilibrium [as stated in equation (11.5.3)] so that

$$\begin{aligned} P(\tau_i) &= \int_0^{\tau_i} g \{ \kappa_0[T(\tau), P_e(\tau)] + \sigma_0[T(\tau), P_e(\tau)] \}^{-1} d\tau \\ &\equiv P[T(\tau_i), P_e(\tau_i)] \end{aligned} \quad (12.2.7)$$

We may look for a value of P_e that makes equations (12.2.6) and (12.2.7) self-consistent. Numerically this can be accomplished by creating tables of κ_0 and σ_0 as functions of P_e so that the integral in equation (12.2.7) can be done directly by any efficient quadrature scheme and then a solution found by iteration with equation (12.2.6). Details of this procedure are given by Mihalas¹. In carrying out this procedure, one keeps the value of the optical depth τ_i sufficiently small that $T(0) \approx T(\tau_i)$. When one has found a self-consistent value of $P_e(\tau_i)$ (and hence all the state variables), values of the state variables may be interpolated for all intermediate optical depths between 0 and τ_i . This technique will provide all the required values of the pressure to initiate a general numerical integration procedure for the differential equation for hydrostatic equilibrium. Since all the other state variables are given in terms of algebraic expressions, the entire atmospheric structure may be obtained as a function of τ_0 .

A note of caution should be interjected at this point concerning the numerical

solution described by this procedure. The range in pressures to be expected from the solution is several powers of 10. For this reason, logarithmic variables are often used to improve the stability of the numerical solution. In any case, the method used to solve the equation of hydrostatic equilibrium should be reasonably sophisticated since the rapid initial growth of the pressure, if not carefully dealt with, can produce systematic errors that destroy the accuracy of the entire atmosphere as the integration proceeds. Several investigators have found it necessary to employ up to a seven-point predictor-corrector integration scheme to achieve the accuracy required. In addition, although the remaining equations are indeed algebraic, the Saha equations for the various elements and states of ionization represent a system of coupled nonlinear algebraic equations and must be solved by iteration. Furthermore, the equations for the opacity due to the different elements in their various states of ionization and excitation represent a significant amount of calculation. Thus, the calculation of $\kappa_{\nu}(\tau_0)$ can be quite time-consuming and represents a significant time burden for the calculation of the model structure. This is particularly evident when one remembers that a multipoint numerical integration scheme requires multiple evaluations of the function g/κ_{ν} to carry out one step in the integration. The situation is further exacerbated by the realization that a rapidly varying numerical solution to a differential equation usually requires that the solution proceed with very small steps, and the range required for the independent variable will be of the order of 2 powers of 10. This is the reason that some modern atmosphere codes utilize a density depth as given in equation (12.2.2) as the independent depth variable. With this choice, the calculation of the opacity is entirely avoided. However, as we see, this choice of an independent variable is not without its own set of problems.

The solution of the equation of hydrostatic equilibrium also provides us with the dependence on depth of all the state variables and the various states of ionization and excitation of the elements. With this information, it is possible to calculate the opacity and hence the radiation field at all points in the atmosphere.

12.3 Calculation of the Radiation Field of the Atmosphere

All Chapter 10 was devoted to solving the equation of radiative transfer, so there is no need to repeat the specifics here. However, some numerical aspects of those solutions require comment. As even the casual reader of Chapter 10 will notice, the general solution of the equation of radiative transfer is fraught with some formidable numerical difficulties. Not the least of these is ensuring the numerical accuracy of the results. Whether one chooses to solve the integrodifferential equation for the specific intensity or the integral equation for the source function of the radiation field, the spacing of the optical depth points at which the solution is to be obtained is crucial for determining the accuracy of that solution.

However, to obtain the radiative flux and source function at a sufficient

number of frequencies to accurately evaluate the condition of radiative equilibrium, we will have to solve the equation of transfer repeatedly over optical depths whose range varies widely. For example, the opacity of a normal stellar atmosphere at frequencies greater than the Lyman limit of hydrogen will be enormously greater than for frequencies in the Balmer continuum for any given physical depth in the atmosphere. Hence, the corresponding optical depth will be very much larger. Since practical realities require that any radiative transfer solution for a semi-infinite atmosphere be truncated at a finite physical depth, we can expect that the monochromatic optical depth corresponding to that physical depth will vary greatly with frequency. However, we must ensure that the radiative transfer solutions, which are calculated at a finite set of depth points that yet span a wide range of optical depths, have sufficient accuracy to facilitate the accurate calculation of radiative equilibrium.

The normal method of accomplishing this is to carry out the numerical solution of the equation of radiative transfer at a predetermined set of optical depths τ_{ri} , chosen to ensure the accuracy of the solution. All physical parameters required for that solution are interpolated from the monochromatic optical depths corresponding to the reference depths onto the set of optical depths to be used for the radiative transfer solution. Mathematically, this amounts to mapping these parameters from the τ_v space on which they are defined onto the τ_r space in which the equation of radiative transfer will be solved. In many cases, the points in the τ_r space can be chosen to be the same points as those used for the reference depth scale τ_0 , but occasionally they may be a separate set of points. In this case, a further mapping of the radiative transfer solution from the τ_r space to the τ_0 space must be carried out. The primary reason for this convoluted procedure is to separate the numerical errors into two well-defined categories - those arising from the interpolation and those arising from the solution of the equation of transfer. The latter are generally more difficult to estimate and so are controlled by carrying out the solution over a set of optical depths for which the numerical stability of the radiative transfer solution is well understood. The errors introduced by the interpolation from one optical depth scale to another are generally easier to control. However, a rapid and accurate mapping algorithm must be available. Such an algorithm is contained in the current version of the atmosphere code known as ATLAS².

For the general overall accuracy of the calculation, we require that the most accurate solutions be obtained at those frequencies for which the majority of the radiative flux flows through the atmosphere. If the reference set of optical depth points is chosen to correspond closely to the monochromatic optical depths at those frequencies, then the interpolation errors incurred from the mapping procedures will be minimal. Frequencies at which the atmosphere is very much optically thicker than the reference optical depths will tend to carry less flux simply because the radiation can escape more easily at the more transparent frequencies. Hence, the frequencies at

which the atmosphere is relatively optically thick, and for which the interpolation errors of the mapping can be expected to be the greatest, will make a correspondingly smaller contribution to the total flux and to the conditions of radiative equilibrium. However, one must be careful that, at the frequencies for which the atmosphere is most transparent, a sufficient number of optical depth points are chosen to ensure that the maximum optical depth is optically remote from the surface. In practice, this generally means that $\tau_\nu \gg 10$.

12.4 Correction of the Temperature Distribution and Radiative Equilibrium

Having created an accurate representation of the radiation field from the previously obtained physical structure, we must see how well the solution conforms to the condition of radiative equilibrium. Departures of the radiation field from that required to satisfy radiative equilibrium will form the basis for correcting the temperature distribution throughout the atmosphere. We have developed the concept of radiative equilibrium several times in this book and most recently in Chapter 10 [equations (10.4.4) and (10.4.5)] as

$$\int_0^\infty \kappa_\nu \rho \{J_\nu(\tau_0) - B_\nu[T_0(\tau_0)]\} d\nu = 0$$

$$\mathbf{F} = \int_0^\infty F_\nu d\nu = \frac{\sigma T_e^4}{\pi}$$
(12.4.1)

Even though these two conditions are logically equivalent, their utilization for generating a temperature correction scheme will be quite different. Although a substantial number of temperature correction schemes have been developed during the last 40 years, we describe only two. The first is chosen for its simplicity and historical interest while the second represents the most widely used method in contemporary use.

a **Lambda Iteration Scheme**

The first of equations (12.4.1) is obtained by setting the total flux derivative to zero. In general, the radiation field obtained from our approximate structure will not satisfy this expression. If we assume that the reason for this is that the temperature used to evaluate the local Planck function is incorrect, we can replace the temperature with a first-order Taylor series expansion about the current temperature. Thus,

$$\begin{aligned}
\int_0^\infty \kappa_\nu J_\nu(\tau_0) d\nu &= \int_0^\infty \kappa_\nu \{B_\nu[T_0(\tau_0) + \delta T]\} d\nu \\
&= \int_0^\infty \kappa_\nu B_\nu[T_0(\tau_0)] d\nu + \delta T \int_0^\infty \kappa_\nu \frac{\partial B_\nu}{\partial T} d\nu
\end{aligned}
\tag{12.4.2}$$

or solving for the temperature correction we have

$$\delta T(\tau_0) = \frac{\int_0^\infty \kappa_\nu [J_\nu(\tau_0) - B_\nu(\tau_0)] d\nu}{\int_0^\infty \kappa_\nu [\partial B_\nu(\tau_0) / \partial T] d\nu}
\tag{12.4.3}$$

This is known as the Λ *iteration scheme* since $J_n(t_0) = \Lambda[B_\nu(\tau_0)]$ [see equation (10.1.16)]. The method yields suitable corrections to the temperature distribution as long as the opacity κ_ν is decidedly nongray. However, as one moves deeper and deeper in the atmosphere, $J_\nu \rightarrow B_\nu$ and the integrand vanishes for all frequencies. Thus, this method relies on the departure of the source function from the value it would have in statistical equilibrium to provide corrections to the local temperature. So while the method may produce a useful temperature correction near the surface, the correction will become smaller and smaller as one descends into the atmosphere. This fact will be reflected in the rate at which the atmosphere converges to a self-consistent value. Indeed, it may become difficult to even know when meaningful convergence has been achieved. To make matters worse, equation (12.4.3) guarantees – in principle – that a self-consistent atmosphere with zero total flux derivative can be calculated. However, it may not have the desired flux, $\sigma T_e^4/\pi$. Thus, we should look for a method for correcting the temperature that employs the second of equations (12.4.1) as well as the first. Such a scheme is due to E. Avrett and M. Krook³ although it is more lucidly described by D. Mihalas¹ (pp. 35 - 39).

b Avrett-Krook Temperature Correction Scheme

Since the temperature correction scheme is to form the basis for an iteration algorithm, it is not essential that it produce the correct temperature the first time it is applied. However, repeated application should produce a series of temperature distributions which approach the one that is correct for radiative equilibrium. Thus, all temperature corrections must vanish asymptotically as the sequence approaches radiative equilibrium. This is the only essential criterion for an iteration scheme. Therefore, it is not necessary to justify all assumptions made in establishing the iterative equations as long as the final result converges to a temperature distribution that is consistent with radiative equilibrium.

The beauty of the Avrett-Krook scheme is that it simultaneously uses both expressions of radiative equilibrium as given in equations 12.4.1. There are two ways of correcting the temperature distribution. The first is the obvious one of simply changing the value of the temperature at some given value of the independent

variable τ_0 . This is the approach taken by the Λ -iteration scheme. A second way to find an improved temperature distribution is to find the value of the independent variable, in this case the reference optical depth, for which the given temperature is the correct temperature. This approach amounts to inverting the problem and treating the temperature as the independent variable and perturbing τ_0 .

The Avrett-Krook scheme does both, using one statement of radiative equilibrium to calculate a temperature correction and the other condition of radiative equilibrium to find a new value of the optical depth at which the *corrected* temperature is to be applied. Thus, both temperature and optical depth become independent variables in the perturbation calculation. The perturbation equations for the temperature are very similar to the Λ -iteration equations and therefore provide good corrections near the surface. The perturbation equations for the optical depth yield small corrections near the surface, but become significant at larger optical depths where the Λ -iteration scheme is ineffective. Thus, the combination yields a temperature correction scheme which converges fairly quickly throughout the entire atmosphere. Unfortunately, the resulting temperature distribution will not directly give the corrected temperatures at the reference optical depths. However, the appropriate temperatures at the reference optical depths can be obtained from the new temperature distribution by interpolation.

The basic approach is to express both the correct temperature and the optical depth in terms of the given values and a first order correction to them, namely,

$$\tau_0 = t + \lambda\tau^{(1)}(t) \quad T(\tau_0) = T^{(0)}(t) + \lambda T^{(1)}(t) \quad (12.4.4)$$

The parameter λ simply measures the order of significance for the particular term and will eventually be set to unity. Substitution of these expressions into the equation of transfer will produce similar corrections in the parameters that describe the radiation field so that

$$\begin{aligned} I_\nu(\mu, \tau_0) &= I_\nu^{(0)}(t, \mu) + \lambda I_\nu^{(1)}(t, \mu) \\ J_\nu(\tau_0) &= J_\nu^{(0)}(t) + \lambda J_\nu^{(1)}(t) \\ F_\nu(\tau_0) &= F_\nu^{(0)}(t) + \lambda F_\nu^{(1)}(t) \end{aligned} \quad (12.4.5)$$

We can expand the normalized opacity [equation (12.2.3)] and the Planck function in a Taylor Series in t and T , respectively, and get

$$\begin{aligned} k_\nu(\tau_0) &= k_\nu^{(0)}(t) + \lambda\tau^{(1)}(t) \frac{dk_\nu(t)}{dt} \\ B_\nu[T(\tau_0)] &= B_\nu[T^{(0)}(t)] + \lambda T^{(1)}(t) \frac{dB_\nu[T^{(0)}(t)]}{dT} \end{aligned} \quad (12.4.6)$$

For simplicity, from now on we denote differentiation with respect to optical depth

and temperature by

$$\frac{d}{dt} = ' \quad \frac{d}{dT} = \cdot \quad (12.4.7)$$

In addition, for clarity we ignore scattering and treat the problem of pure absorption only. Later we give the perturbation equations appropriate for a source function that includes scattering, justifying the results on physical grounds alone.

Perturbed Equation of Radiative Transfer The general nongray equation of transfer for a plane-parallel atmosphere for the case of pure absorption is

$$\mu \frac{dI(\mu, \tau_0)}{d\tau_0} = k_v \{ I_v(\mu, \tau_0) - B_v[T(\tau_0)] \} \quad (12.4.8)$$

If we insert the expansions given by equations (12.4.4) and (12.4.5) into this equation and ignore all second order terms (i.e., terms involving λ^2), we get

$$\begin{aligned} \mu \frac{dI_v^{(0)}}{dt} - \lambda \mu \tau^{(1)} \frac{dI_v^{(0)}}{dt} + \mu \lambda \frac{dI_v^{(1)}}{dt} &= (k_v^{(0)} + \lambda \tau^{(1)} k_v^{\prime(0)})(I_v^{(0)} - B_v^{(0)}) \\ &+ \lambda k_v^{(0)}(I_v^{(1)} + T^{(1)} \dot{B}_v^{(0)}) \end{aligned} \quad (12.4.9)$$

Since this equation must hold for any value of λ , we can separate the zeroth- and first-order terms. The zeroth-order equation is then

$$\mu \frac{dI_v^{(0)}}{dt} = k_v^{(0)}(I_v^{(0)} - B_v^{(0)}) \quad (12.4.10)$$

We can use this result to eliminate $dI_v^{(0)}/dt$ from the first-order equation so that it becomes

$$\mu \frac{dI_v^{(1)}}{dt} = k_v^{(0)}(I_v^{(1)} + T^{(1)} \dot{B}_v^{(0)}) + (\tau^{(1)} k_v^{(0)} + \tau^{(1)} k_v^{\prime(0)})(I_v^{(0)} - B_v^{(0)}) \quad (12.4.11)$$

This equation can be solved by using the Eddington approximation to moments of the equation in a manner that should be familiar by now.

Forming the first two moments of equation (12.4.11) (i.e., just integrating over all μ to obtain the first and multiplying by μ and integrating to get the second), we obtain

$$\begin{aligned} F^{(1)} &= k_v^{(0)}(J_v^{(1)} + T^{(1)} \dot{B}_v^{(0)}) + (\tau^{(1)} k_v^{(0)} + \tau^{(1)} k_v^{\prime(0)})(J_v^{(0)} - B_v^{(0)}) \\ K_v^{(1)} &= \frac{J_v^{(1)}}{3} = k_v^{(0)} F_v^{(1)} + (\tau^{(1)} k_v^{(0)} + \tau^{(1)} k_v^{\prime(0)}) F_v^{(0)} \end{aligned} \quad (12.4.12)$$

In the second equation, we have already assumed that the Eddington approximation can be applied to the first-order perturbations as it is to the entire radiation field.

Tau Perturbation Equation Now we integrate the second of equations (12.4.12) over all frequencies and get

$$\frac{1}{3} \int_0^\infty \frac{J'_v{}^{(1)}}{k_v^{(0)}} dv = \int_0^\infty F_v^{(1)} dv + \tau^{(1)} \int_0^\infty F_v^{(0)} dv + \tau^{(1)} \int_0^\infty \frac{k'_v{}^{(0)}}{k_v^{(0)}} F_v^{(0)} dv \quad (12.4.13)$$

Requiring that

$$J'_v{}^{(1)}(t) = 0 \quad (12.4.14)$$

guarantees that the left-hand side of equation (12.4.13) will vanish. The assumption stated by equation (12.4.14) is justified by expediency alone. However, it is an assumption concerning the perturbation only and therefore can affect only the rate of convergence. There may be some instances where this approximation should be replaced. However, to do so, we must know something additional about the problem.

The first term on the right-hand side of equation (12.4.13) is just the integrated flux error so that

$$\int_0^\infty F_v^{(1)}(t) dv = \mathbf{F} - F^{(0)} \quad (12.4.15)$$

where

$$\mathbf{F} = \frac{\sigma T_e^4}{\pi} \quad (12.4.16)$$

With this, we can rewrite equation (12.4.12) as a first-order linear differential equation for the perturbed optical depth

$$\tau^{(1)} + \tau^{(1)} \int_0^\infty \frac{k'_v{}^{(0)} F_v^{(0)}}{k_v^{(0)} F^{(0)}} dv = 1 - \frac{\mathbf{F}}{F^{(0)}} \quad (12.4.17)$$

All that remains is to specify a boundary condition for the solution of the equation. An appropriate condition is

$$\tau^{(1)}(0) = 0 \quad (12.4.18)$$

While this condition appears to be arbitrary, it anticipates the result for the T perturbation which will provide the majority of the correction at the surface. The

boundary condition given in equation (12.4.18) will ensure that the tau corrections are small near the surface and thus will not compete heavily with the T corrections.

Temperature Perturbation Equation To obtain the T perturbation equation, we begin with the first of equations (12.4.12). Since we required that the derivative of the perturbed mean intensity $J_\nu^{(1)}$ be zero at all frequencies and depths [equation (12.4.14)], we may get the last term on the right-hand side of the first of equations (12.4.12) from the second equation, so that

$$\tau^{(1)}k_\nu^{(0)} + \tau^{(1)}k_\nu^{\prime(0)} = -\frac{k_\nu^{(0)}F_\nu^{(1)}}{F_\nu^{(0)}} \quad (12.4.19)$$

That same assumption on the derivative of the perturbed mean intensity will require that

$$J_\nu^{(1)}(t) = \text{const} = J_\nu^{(1)}(0) = aF_\nu^{(1)}(0) \quad (12.4.20)$$

The last term implies the Eddington approximation; so that a is usually taken to be $\frac{1}{2}$. However, some authors use somewhat different values for a based on empirical work. As with any iteration scheme, one that works is a good one. Remembering that we have assumed a boundary condition on the $\tau^{(1)}$ - equation of $\tau^{(1)} = 0$, we see that equations (12.4.17) and (12.4.19) give

$$\tau^{(1)}(0) = -\frac{F_\nu^{(1)}(0)}{F_\nu^{(0)}(0)} = 1 - \frac{\mathbf{F}}{F^{(0)}(0)} \quad (12.4.21)$$

Thus, we may obtain the perturbed value for J as

$$J_\nu^{(1)}(t) = -aF_\nu^{(0)}(0) \left[1 - \frac{\mathbf{F}}{F^{(0)}(0)} \right] = \text{const} \quad (12.4.22)$$

Inserting this result and equation (12.4.19) into the first of equations (12.4.12) we get

$$\begin{aligned} F_\nu^{\prime(1)}(t) &= k_\nu^{(0)}(t) \left\{ aF_\nu^{(0)}(0) \left[1 - \frac{\mathbf{F}}{F^{(0)}(0)} \right] + T^{(1)}(t)\dot{B}_\nu^{(0)} \right\} \\ &= -\frac{k_\nu^{(0)}F_\nu^{(1)}}{F_\nu^{(0)}} [J_\nu^{(0)}(t) - B_\nu^{(0)}] \end{aligned} \quad (12.4.23)$$

From the definition of $F_\nu^{(1)}$ we know that

$$F_\nu^{\prime(1)}(t) = \frac{d}{dt} \left[\mathbf{F} - F_\nu^{(0)}(t) \right] = -F_\nu^{\prime(0)} \quad (12.4.24)$$

Incorporating this into equation (12.4.23), integrating over all frequencies, and remembering that the condition of radiative equilibrium applies to the zeroth-order equations, we finally get the perturbation equation for the temperature as

$$T^{(1)}(t) = \frac{\mathbf{F} \int_0^\infty k_\nu^{(0)}(t) \{a[1 - F_\nu^{(0)}(0)/\mathbf{F}] + [J_\nu^{(0)}(t) - B_\nu^{(0)}(t)]/F_\nu^{(0)}(t)\} d\nu}{\int_0^\infty \dot{B}_\nu^{(0)}(t) k_\nu^{(0)}(t) d\nu} \quad (12.4.25)$$

We now have expressions for the temperature corrections $T^{(1)}(t)$ and the corrected values of the optical depth $t + \tau^{(1)}$, to which they are to apply. Interpolation of this temperature distribution back onto the original optical depth scale completes the temperature correction procedure. A comparison of equations (12.4.25) and (12.4.3) shows that the Avrett-Krook temperature correction equation is indeed very close to the Λ -iteration equation. However, an additive constant appears in the Avrett-Krook equation which ensures that the corrections will converge to the correct flux \mathbf{F} .

Perturbation Equations Including Scattering The inclusion of scattering significantly complicates the algebra of deriving the perturbation equations, but not the concept. However, the essence of the problem can be seen without suffering through the algebra of the derivation. Consider a very general source function such as that given in equation (10.1.7). The parameter ϵ_ν is a measure of the fraction of photon interactions that can be viewed as pure absorptions. Thus, $1 - \epsilon_\nu$ is the relative fraction of scatterings. Since at the microscopic level scattering is a fully conservative process, we should expect it to have no influence on the physical structure of the atmosphere. Scattering decouples the radiation field from the physical domain of the gas. Thus, any temperature correction procedure will become less well defined for an atmosphere where the opacity becomes more nearly gray.

To carry out the perturbation analysis, we must add a perturbation equation for $\epsilon_\nu(\tau_0)$ similar to equations (12.4.6). It could take the form

$$\epsilon_\nu(\tau_0) = \epsilon_\nu^{(0)}(t) + \lambda \tau^{(1)}(t) \epsilon_\nu^{\prime(0)}(t) \quad (12.4.26)$$

As with the opacity, an assumption is made that the derivatives with respect to optical depth are more important than the derivatives with respect to temperature. The appropriate equation of radiative transfer analogous to equation (12.4.8) is then

$$\mu \frac{dI(\mu, \tau_0)}{d\tau_0} = k_\nu \{I_\nu(\mu, \tau_0) - \epsilon_\nu(\tau_0) B_\nu[T(\tau_0)] - [1 - \epsilon_\nu(\tau_0)] J_\nu(\tau_0)\} \quad (12.4.27)$$

where κ_ν is now defined by

$$k_\nu(\tau_0) \equiv \frac{\kappa_\nu(\tau_0) + \sigma_\nu(\tau_0)}{\kappa_0(\tau_0) + \sigma_0(\tau_0)} \quad (12.4.28)$$

Development of the two moment equations analogous to equations (12.4.12) will show that the second is unchanged by the presence of scattering. This leads to the happy result that the tau perturbation equation is also unchanged, so that equation (12.4.17) and its solution are correct for the more general case including scattering.

The presence of scattering does modify the first moment equation of equations (12.4.12). This yields a somewhat different temperature perturbation equation from equation (12.4.25). With scattering, it takes the form

$$T^{(1)}(t) = \frac{\mathbf{F} \int_0^\infty k_\nu^{(0)}(t) \{ a\epsilon_\nu^{(0)}(t) [1 - F_\nu^{(0)}(0)/\mathbf{F}] + [J_\nu^{(0)}(t) - B_\nu^{(0)}(t)] \times [\tau^{(1)}(t)\epsilon_\nu^{(0)}(t) + \epsilon_\nu^{(0)}(t)\mathbf{F}/F_\nu^{(0)}(t)]/\mathbf{F} \} d\nu}{\int_0^\infty \dot{B}_\nu^{(0)}(t)k_\nu^{(0)}(t)\epsilon_\nu^{(0)}(t) d\nu} \quad (12.4.29)$$

In the limit of pure absorption where $\epsilon_\nu \rightarrow 1$, we recover immediately equation (12.4.25). As we approach the limit of a pure scattering atmosphere $\epsilon_\nu \rightarrow 0$. All terms in the numerator of equation (12.4.29) clearly vanish. Unfortunately so does the denominator, leaving the asymptotic behavior of $T^{(1)}$ in doubt. An application of L'Hospital's rule shows that the temperature correction terms indeed formally go to zero for the case of pure scattering. However, many of the terms of equation (12.4.29) are difficult to calculate numerically so that the practical result of increased scattering will be to at first slow the rate of convergence of the iteration procedure. The iteration procedure will become unstable as the amount of scattering becomes very large. This is not surprising since the instability merely reflects the decoupling of the radiation field from the physical structure of the atmosphere.

Equations (12.4.17) and (12.4.29) provide the mechanism by which departures from radiative equilibrium can be translated to an improved temperature distribution. With this temperature distribution, we may return to the beginning of this chapter and re-compute the structure and improved radiation field of the atmosphere. The entire process can be iterated until radiative equilibrium is satisfied at the appropriate level.

12.5 Recapitulation

Building on the results of the previous three chapters, We present in this chapter the basic approach to the construction of a model stellar atmosphere. The process is essentially an iterative one where an initial guess of the temperature distribution throughout the atmosphere yields the atmosphere's physical structure. To obtain this structure, one needs a lot of information about the dependence of the opacity on the state variables of the gas. One generally assumes that the Saha ionization and Boltzmann excitation formulas hold so that one can relatively easily calculate the abundance of each type of absorber in the atmosphere. This, then, allows for the

solution of the equation of hydrostatic equilibrium and the calculation of the radiation field at all points in the atmosphere. Application of the condition of radiative equilibrium and a temperature correction procedure allow for the entire process to be iterated until a self-consistent model of the atmosphere is obtained.

What constitutes a converged, self-consistent atmosphere is not at all obvious and may depend on which properties of the model are of particular interest to the investigator. For example, it makes little sense to require 0.1 percent constancy in the radiative flux at optical depth 100 if one is interested in only the emergent flux. Conversely, if the atmosphere is to form the boundary layer for the calculation of a model stellar interior, then some care should be taken with the deep solution. If one is concerned about strong spectral lines, then considerable care should be taken with the surface solution. Since it is still relatively difficult to construct a model stellar atmosphere that exhibits both a constant flux and a zero flux derivative throughout the atmosphere at an arbitrary level of accuracy, such considerations regarding the use of the model should be weighed.

We have now completed the fundamental physics concerning the construction of models for both the inside and surface layers of a star. For normal stars, these models would give a reasonably accurate picture of the structure of these stars and the processes that take place within them. We have even included the departure of the radiation field from strict thermodynamic equilibrium that results from the escape of photons that are near the surface into free space. To the extent that the continuous opacity dominates the total stellar opacity, this will even yield a reasonably correct picture of the grosser aspects of the star's spectrum. However, as anyone who has looked at a stellar spectrum knows, the most salient feature of such a spectrum is the dark lines that cover it. These lines provide most of the information that we have about stars, from their composition to their motions. No description of stellar structure can hope to be taken seriously unless it provides some explanation of the occurrence of these lines. Therefore, for the majority of the rest of this book, we discuss the fundamentals of the formation of spectral absorption lines and the physics that yields their characteristic shape.

Problems

1. Assume that all particles in a normal GV star at optical depth $\tau(\lambda 5000) = 1$ have the same speed. Estimate the time required for LTE to be established.
2. Starting with the gray atmosphere temperature distribution, find the rate of convergence toward radiative equilibrium as a function of the Rosseland optical depth for a standard atmosphere (i.e., $T_e = 10,000$ K, $\text{Log } g = 4.0$, and the chemical composition m equals that of the sun). Explicitly define what *you* mean by the "rate of convergence".

3. As one moves deeper into a stellar atmosphere, the dependence of the source function becomes more linear with optical depth. Is this a result of the opacity becoming more gray (i.e., independent of wavelength), or does the result follow from the directional randomization of the radiation field? Give explicit evidence to support your conclusion.
4. Compute $F_\lambda/F_\lambda(\lambda 5560)$ for a nongray atmosphere where
 - (a) $\sigma_\lambda = 0$, and $\kappa_\lambda = a + b\lambda$ and the effective temperature $T_e = 5000$ K, and
 - (b) same as in (a) but with $\kappa_\lambda = a$.

Assume that the Eddington approximation is sufficiently accurate to solve the equation of radiative transfer.

- (c) how do $F_\lambda(12000)$ and $F_\lambda(15560)$ vary with the optical depth.

References and Supplemental Reading

1. Mihalas, D. *Methods in Computational Physics*, (Eds: B. Alder, S. Fernbach, and M. Rotenberg), Vol. 7 Academic, New York, 1967, pp.24 - 27.
- 2.. Kurucz, R.L. *ATLAS: A Computer Program for Calculating Model Stellar Atmospheres*, SAO Special Report 309, 1970.
4. Avrett, E.H., and Krook, M. *The Temperature Distribution in a Stellar Atmosphere*, Ap.J.137, 1963, pp.874 - 880.

For further insight into temperature correction procedures, the student should read

Mihalas, D.: *Stellar Atmospheres*, W.H. Freeman, San Francisco, 1970
pp.169 - 186.

Additional Reading on the Subject of Stellar Atmospheres

Often some of the oldest references remain the most illuminating in any subject. This is particularly true of stellar atmospheres. The computer has made it too easy to avoid thinking about the interaction of the physical processes going on in the atmosphere and to rely instead on the output of the machine. Those references that predate the computer often focus on this physics, for the authors had no other option. Since knowledge and understanding of these interactions are central to the understanding of stellar atmospheres, we strongly recommend that the serious student make some effort to at least peruse some of these references:

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