5

Motion under the Influence of a Central Force

The fundamental forces of nature depend only on the distance from the source. All the complex interactions that occur in the real world arise from these forces, and while many of them are usually described in a more complex manner, their origin can be found in the fundamental forces that depend only on distance. Thus even the intricate forces of aerodynamic drag can ultimately be described as resulting from the electrostatic potential of the air molecules scattering with the electrostatic potential of the molecules of the aircraft, and the electrostatic force is essentially one that depends on distance alone. It is the presence of many sources of the distance-dependent forces that enables the complex world we know to exist. Thus, in order to understand complex phenomena, it is appropriate that we begin with the simplest. Therefore we will begin by applying the tools of mechanics developed in the previous two chapters to describe the motion of an object moving under the influence of a single source of a force that depends only on the distance. We will call this object a "test particle" to make clear that its motion in no way affects the source of the potential. Such a situation is known as a central force problem since the source may be located at the origin of the coordinate system making it central to the resulting description.

5.1 Symmetry, Conservation Laws, the Lagrangian, and Hamiltonian for Central Forces

Since there is a single source producing a force that depends only on distance, the force law is spherically symmetric. If this is the case, then there can be no torques present in the system as there would have to be a preferred axis about which the torques act. That would violate the spherical symmetry so

$$\vec{N} = \frac{d\vec{L}}{dt} = 0 \quad . \tag{5.1.1}$$

Equation (5.1.1) clearly means that the total angular momentum of the test particle does not change in time. Specifically, it means that the direction of the angular momentum vector doesn't change. Since there is only one particle in this system, this is little more than a statement of the conservation of angular momentum, but it has a great simplifying implication. The radius vector $\vec{r}$ and the particle's linear momentum $\vec{p}$ define a plane. Since

$$\vec{L} = \vec{r} \times \vec{p} \quad , \tag{5.1.2}$$

the angular momentum is always perpendicular to that plane and being constant in space requires that the motion of the particle is confined to that plane. Thus we can immediately reduce the problem to a two dimensional description.

Since there is only one particle in the system and we require the total energy of the system to be constant, the total energy of the particle must be constant. Thus such a force is conservative and we may use the Lagrangian formalism of Chapter 3 to obtain the equations of motion. We begin this procedure by choosing a set of generalized coordinates. Remember that the only requirement for the generalized coordinates is that they span the space of the motion and be linearly independent. For motion that is confined to a plane defined by the action of a central force, the logical choice of a coordinate frame is polar coordinates with the center of the force field located at the origin of the coordinate system. However, since the kinetic energy is more obviously written in Cartesian coordinates, let us use the definition of the Lagrangian to write

$$\mathscr{L} = T - V = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - m\Phi(r) \quad , \tag{5.1.3}$$

where $\Phi(r)$ is the potential giving rise to the conservative central force. The transformation from Cartesian coordinates to polar coordinates is

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \quad , \tag{5.1.4}$$

so that

$$\left.\begin{aligned}\dot{x}^2 &= \dot{r}^2\cos^2\theta - 2r\dot{r}\sin\theta\cos\theta\,\dot{\theta} + r^2\dot{\theta}^2\,\sin^2\theta \\ \dot{y}^2 &= \dot{r}^2\sin^2\theta - 2r\dot{r}\sin\theta\cos\theta\,\dot{\theta} + r^2\dot{\theta}^2\,\cos^2\theta\end{aligned}\right\} \quad . \tag{5.1.5}$$

Substitution of these expressions into the Lagrangian gives

$$\mathscr{L} = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - m\Phi(r) \quad . \tag{5.1.6}$$

Lagrange's equations of motion for polar coordinates will then be

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{r}}\right) - \frac{\partial\mathscr{L}}{\partial r} = 0 \\ \frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathscr{L}}{\partial\theta} = 0\end{aligned}\right\} \quad . \tag{5.1.7}$$

In terms of the polar coordinates [r,θ] the quantities required for Lagrange's equations of motion are:

$$\left.\begin{aligned}\frac{\partial\mathscr{L}}{\partial\dot{r}} &= m\dot{r} \\ \frac{\partial\mathscr{L}}{\partial\dot{\theta}} &= mr^2\dot{\theta} \\ \frac{\partial\mathscr{L}}{\partial r} &= mr\dot{\theta}^2 - m\frac{\partial\Phi(r)}{\partial r} \\ \frac{\partial\mathscr{L}}{\partial\theta} &= 0\end{aligned}\right\} \quad . \tag{5.1.8}$$

so that the explicit equations of motion become

$$\left.\begin{aligned}m\ddot{r} - mr\dot{\theta}^2 + m\frac{\partial\Phi(r)}{\partial r} = 0 \\ \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0\end{aligned}\right\} \quad . \tag{5.1.9}$$

Now in Chapter 3 we developed the Hamiltonian from the Lagrangian and the generalized momenta [see equations (3.3.1- 3.3.3)] and it is illustrative to do this explicitly for the case of a central force. The generalized momenta can be obtained from the Lagrangian by means of equation (3.3.1) so for polar coordinates we have

$$\left.\begin{aligned} q_i &= [r,\ \theta] \\ p_i &= [m\dot{r},\ mr^2\dot{\theta}] \end{aligned}\right\} \quad . \tag{5.1.10}$$

From equation (3.3.3) we can then write the Hamiltonian as

$$H(p_r, p_\theta, r, \theta, t) = \left(m\dot{r}^2 + mr^2\dot{\theta}^2\right) - \left(\tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2\right) + m\Phi(r) = T + U = E \ . \tag{5.1.11}$$

As long as the Lagrangian and generalized coordinates were not explicit functions of time, the Hamiltonian is an integral of the motion. Since from equation (5.1.11) it is clear that the Hamiltonian is also the total energy, we have effectively recovered the law of conservation of energy. The Hamilton canonical equations of motion [see equations (3.3.6)] effectively add nothing new to the problem since we already have two constants of the motion (i.e., the angular momentum and the total energy). Thus we can turn to the implications of these two constants.

5.2 The Areal Velocity and Kepler's Second Law

The θ-equation of motion that gives us the constancy of angular momentum enables us to write

$$mr^2\dot{\theta} = mr^2\omega = L \quad . \tag{5.2.1}$$

The differential area included between two radius vectors separated by a differential angle dθ is just

$$dA = \tfrac{1}{2}r(rd\theta) \ . \tag{5.2.2}$$

Let us call the time derivative of this area the areal velocity so that

$$\frac{dA}{dt} = \tfrac{1}{2}r^2\frac{d\theta}{dt} = \tfrac{1}{2}\frac{L}{m} = \text{const.} \tag{5.2.3}$$

Thus we could say that the areal velocity of the test particle is constant and merely be re-stating the conservation of angular momentum. This is indeed the way Johannes Kepler gave his second law of planetary motion. However, Kepler had no conception of angular momentum and his laws dealt only with the planets. Here we see that Kepler's second law not only applies to objects moving under the influence of the gravitational force, but will hold for an object moving under the influence of any central force regardless of its distance dependence. Thus Kepler's second law of planetary motion is far more general than Kepler ever knew.

We may use this result to eliminate $\dot{\theta}$ from the first of the two Lagrangian equations of motion and thereby reduce the problem to that of one dimension. Solving equation (5.2.1) for $\dot{\theta}$ and substituting into the r-equation (5.1.9) we get

$$m\ddot{r} - \frac{L^2}{mr^3} + \frac{\partial\Phi(r)}{\partial r} = 0 \quad . \tag{5.2.4}$$

We can obtain the result of the Hamiltonian directly by multiplying equation (5.2.4) by $\dot{r}$ and re-writing as

$$\frac{d}{dt}\left[\tfrac{1}{2}m\dot{r}^2\right] = -\frac{d}{dt}\left[\Phi(r) + \frac{L^2}{2mr^2}\right]m = -\frac{d}{dt}\left[\Phi(r) + \tfrac{1}{2}r^2\dot{\theta}^2\right]m \quad , \tag{5.2.5}$$

which becomes

$$\frac{d}{dt}\left[\tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + m\Phi(r)\right] = \frac{d}{dt}\left(T + V\right) = \frac{dE}{dt} = 0 \quad . \tag{5.2.6}$$

5.3 The Solution of the Equations of Motion

Finding the two integrals of the motion goes along way to completing the solution of the problem. These two constants essentially amount to integrating each of equations (5.1.9) once so that the equations of motion can be written as

$$\left.\begin{array}{l} \tfrac{1}{2}m\dot{r}^2 + \left[\dfrac{L^2}{(2mr^2)} + m\Phi(r)\right] = E = \text{const.} \\ mr^2\dot{\theta} = L = \text{const.} \end{array}\right\} \quad . \tag{5.3.1}$$

The first of these is obtained by integrating equation (5.2.6) and replacing $\dot{\theta}$ from equation (5.2.1). The other is equation (5.2.1) itself. These are two first order differential equations and require two more constants to completely specify their solution. These two additional constants are known as initial conditions and it is worth distinguishing them from integrals of the motion. An *integral of the motion* will have the same value for the entire temporal history of the system while an *initial value or boundary condition* is just the value of one of the dependent variables of the problem at some *specific* instant in time. Surely the latter is a constant since it is specified at a given time, but an integral of the motion is some combination of the dependent variables that is constant for all time. Integrals of the motions are exceedingly important to any dynamics problem and knowledge of them, as we shall see later, places very useful constraints on the history and nature of the system.

The solution of the second of equations (5.3.1) yields the temporal history of the variable θ and can be obtained by direct integration of that equation so that

$$\theta(t) = \int_0^t \frac{L\,dt}{mr^2(t)} + \theta_0 \quad . \tag{5.3.2}$$

Here the constant θ_0 is the initial value of θ at t = 0. We have chosen the initial value of the time to be zero, but that is arbitrary and the initial value could have been anything. The first of equations (5.3.1) is somewhat more difficult to solve. Direct integration gives

$$t(r) = \left(\frac{m}{2}\right)^{1/2} \int_{r_0}^{r} \frac{dr}{[E - \Phi(r) - L/2mr^2]^{1/2}} \quad . \tag{5.3.3}$$

Thus the r-coordinate is given as an implicit function of time t(r). This implicit function must be inverted to have the same form as equation (5.3.2). The parameter r_0 is the second initial value, being the value of r at t = 0. Thus the two initial values $[\theta_0, r_0]$, and the integrals of the motion [L,E] completely specify the motion of the particle. However, to solve a specific problem we must specify Φ(r) because the integral, and subsequent inversion of equation (5.3.3), cannot be done without knowledge of Φ(r).

In any event, we may put rather general limits on the range of solutions that we can expect for any given Φ(r). The second of equations (5.3.3) is essentially a one dimensional equation in r, so we will define anew potential

$$\varnothing(r) = \Phi(r) + \frac{L^2}{2m^2r^2} \quad . \tag{5.3.4}$$

We can require that the potential energy vanish as $r \rightarrow \infty$ as a reasonable boundary condition. Thus the kinetic energy is

$$T = E - m\varnothing(r) \ . \tag{5.3.5}$$

Now if $\varnothing(r) \geq 0$ then the force law is repulsive, and there is some minimum distance r_{min} to which the particle can approach the source before the right hand side of equation (5.3.5) would become negative, implying a nonphysical negative kinetic energy. This is a plausible result that simply says that a repulsive force which increases in strength as its source is approached will eventually stop the approach. The total energy for such a system must also always be greater than zero.

For the more interesting case of power law potentials where $\varnothing(r) \leq 0$, we can write

$$\varnothing(r) = \left[\tfrac{1}{2}\frac{L^2}{m} - \frac{k}{r^{(n-2)}} \right] \Big/ r^2 \quad , \tag{5.3.6}$$

where k is some positive constant. Interesting things will happen at

$$r_0 = \left(2mk/L^2\right)^{[1/(n-2)]} \quad . \tag{5.3.7}$$

If power law dependence is such that n > 2, then r_0 serves as an upper bound for particles with a total energy E < 0.

For the more pertinent case of gravity where n = 1, then r_0 serves as a lower bound inside of which particles may not approach. The physical interpretation of this lower bound is simple. If L is not zero, then there is some angular motion of the particle as it orbits the central source. However, as it approaches the central source, the conservation of angular momentum will require an increase in its angular velocity keeping the particle from approaching closer. Colloquially one could say that the particle is repelled by centrifugal force. Indeed, this part of the pseudo-potential $\varnothing$ is often known as the "rotational potential". Thus, if the total angular momentum L > 0, then the motion of the particle will be kept beyond r_0 . If the total energy E < 0, then there will also be an upper bound r_l since the kinetic energy must remain positive. Thus for cases in which the total energy E < 0, the particle's motion will be confined between $r_l \geq r \geq r_0$. This is the case for virtually all motion in the solar system.

5.4 The Orbit Equation and Its Solution for the Gravitational Force

To see more clearly the solution to the equations of motion, let us eliminate time as the independent variable from equations (5.1.9). Thus, we search for a single equation involving r as a function of θ. We can do this by noting that the total time derivative operator can be written as

$$\frac{d}{dt}=\frac{d\theta}{dt}\frac{d}{d\theta}=\frac{L}{mr^2}\frac{d}{d\theta} \quad , \tag{5.4.1}$$

and

$$\frac{1}{r^2}\frac{dr}{d\theta}=-\frac{d(1/r)}{d\theta} \quad . \tag{5.4.2}$$

Thus, replacing the time derivatives of equation (5.1.9) by equation (5.4.1) and using equation (5.4.2) we can write

$$\frac{-L^2}{mr^2}\frac{d^2(1/r)}{d\theta^2}-\frac{L^2}{mr^3}=-m\frac{\partial\Phi}{\partial r}\equiv f(r) \quad . \tag{5.4.3}$$

This is the so called orbit equation since its solution is r(θ), the orbit of the particle. This equation is more amenable to solution if we re-write it by substituting

$$u\equiv\frac{1}{r} \quad , \tag{5.4.4}$$

so that

$$\frac{L^2u^2}{m}\left(\frac{d^2u}{d\theta^2}+u\right)=-f(1/u) \quad . \tag{5.4.5}$$

The quantity f(r) or f(l/u) is the force law, which for gravity is

$$f(r)=-\frac{GMm}{r^2}=-GMmu^2 \quad . \tag{5.4.6}$$

Thus the orbit equation for the gravitational force takes the following relatively simple form:

$$\frac{d^2u}{d\theta^2}+u=\frac{GMm^2}{L^2}\equiv\beta=\text{const.} \tag{5.4.7}$$

This is a second order equation so we can expect two constants of integration in the most general solution. As is customary with differential equations of this form, we can guess a solution to be

$$u=A\cos(\theta-\theta_0)+\beta \quad , \tag{5.4.8}$$

which we can re-write as

$$r = \frac{P}{[1 + e\cos(\theta - \theta_0)]} \quad , \tag{5.4.9}$$

so long as

$$\left. \begin{aligned} P &= \frac{1}{\beta} = \frac{L^2}{GMm^2} \\ e &= \frac{AL^2}{GMm^2} \end{aligned} \right\} \quad . \tag{5.4.10}$$

The reason for this last transformation is that equation (5.4.9) is the general equation for a conic section with a focus at the coordinate origin and an eccentricity e. Thus we recover the essence of Kepler's first law, namely that the planets move in ellipses with the sun at one focus. Certainly ellipses are conic sections, and should the mass of the sun greatly exceed the mass of the earth, then the sun may be regarded as the source of the central force of gravity. We have now only to decide which type of conic section the orbit will be and on what parameter the kind will depend.

We can make this determination by generating the total energy from the solution to the orbit equation. First differentiate the solution with respect to time and after some algebra find the velocity to be given by

$$v^2 = \left(L^2/m^2P^2\right)\left[\left(e^2 - 1\right) + \left(2P/r\right)\right] \quad . \tag{5.4.11}$$

Now form the kinetic energy and add the potential energy to get

$$E = \tfrac{1}{2}\left(GMm/P\right)\left(e^2 1\right) \quad . \tag{5.4.12}$$

It is clear from equation (5.4.12) that the sign of the total energy will determine the sign of the eccentricity e and hence the type of orbit namely

$$\left. \begin{aligned} E > 0 &\Rightarrow e^2 > 1 \ : \ \text{hyperbolic orbit} \\ E = 0 &\Rightarrow e^2 = 1 \ : \ \text{parabolic orbit} \\ E < 0 &\Rightarrow e^2 < 1 \ : \ \text{elliptic orbit} \end{aligned} \right\} \quad . \tag{5.4.13}$$

Thus we see that Kepler's first law of planetary motion implies that the total orbital energy of the planets is negative so that the planets are bound. In addition, any bound test particle in orbit about the sun will have an elliptic orbit.

Chapter 5: Exercises

1. Given that a particle is moving under the influence of a central force of the form

$$f = -\frac{k}{r^2} + \frac{c}{r^3} \quad ,$$

where k and c are positive constants. Show that the solution to the orbit equation can be put in the form

$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos(\alpha\theta)} \quad ,$$

which is an ellipse for $e < 1$ and $\alpha = 1$. Discuss the character of the orbit for $\alpha \neq 1$ and $e < 1$. Derive an approximate expression for α in terms of the dimensionless parameter $\gamma = [c/(ka)]$.

2. Discuss the motion of a test particle moving in a potential field of the form

$$\Phi(r) = (\alpha/r) + (\beta/r^3) \quad ,$$

in terms of the rotational potential and conservation of energy.

3. A particle moves in a circular orbit of radius r_0 under the influence of a central force located at some point inside the orbit. The minimum and maximum speeds of the particle are v_1 and v_2 respectively. Find the orbital period in terms of these speeds and the radius of the orbit.

4. Suppose that all the planets move about the sun in circular orbits under the influence of an inverse cube force law. Assuming conservation of momentum and energy, find a relation between the orbital period and the radius of the orbit for the planets. (i.e. a new "Kepler's third law").