8

The Dynamics of More Than Two Bodies

In Chapter 3 we established the general principles of Newtonian mechanics and the mathematical formalism for the determination of the equations of motion for the objects that make up an arbitrary mechanical system. We used those principles in Chapter 5 to describe the motion of two bodies under their mutual gravitational attraction. As we shall see, problems dealing with more than two bodies become extremely complicated and do not, in general, yield closed form solutions. The dynamical behavior of large systems of stars that seem to populate the central regions of galaxies is currently a problem of intense study three and a half centuries after Newton identified the principles that guide their motion. Before we even attempt to discuss systems consisting of a large number of objects, we shall discuss systems of three objects.

8.1 The Restricted Three Body Problem

Certainly the next logical step after the solution of the two body problem is the addition of a third body. Yet even here we find that the general problem is unsolved. Nature seems to deal with the problem in a simple manner for there are many stellar systems consisting of three or more stars bound by their mutual gravitational attraction. However, in all of these systems, the objects seem to degenerate to a hierarchical succession of two body problems. For example, should the system contain three stars, two will be tightly bound orbiting as one would expect from the two body solution and the third will be found at a distance corresponding to many times the separation of the close orbiting pair. Four gravitationally bound stars always appear as a binary of binaries and so forth. It is generally believed that there are no stable orbits involving three comparable masses with comparable separations.

The source of the difficulty in dealing with as few as three objects can be found in the notion of the integrals of the motion discussed in Chapter 5 (Sec 5.3). In any mechanics problem one always has the Hamiltonian, or the total energy, and the total angular momentum as integrals of the motion. These are quantities which will be constant throughout the motion of the members of the system wherever they may go. Since the angular momentum is a vector quantity, it has three linearly independent components, each of which serves as a constant of the motion. Thus the conservation of energy and angular momentum provide four constants that restrict the motion of the system. Taken together with the six constants that specify the uniform motion of the center of mass, there remain only two constants to completely determine the motion of a two body system. It is the quadratic nature of the force law that requires that solutions for the orbits will also be quadratic and thus if the orbits are bound they will be closed. This is not the case for other force laws, as is evidenced by the precession of the perihelion of Mercury's orbit resulting from the presence of masses other than the sun in the solar system. Mercury's orbit isn't quite elliptical and never exactly closes in space. Closure requires that the object return to the same physical location with the same velocity. Thus the last constant serves only to locate the particle in its orbit.

Since the general problem of three bodies will be described by a second order vector differential equation for each of the particles, there will be l8 constants of motion. The conservation of angular momentum and energy together with the uniform motion of the center of mass will provide 10 constants leaving eight to be determined. Since the general potential affecting anyone of the objects will not be that of a single point mass we should not expect the orbits of the objects to close and we are left with eight arbitrary constants required to specify the problem. Thus the motion is in no way uniquely determined by the conservation laws of physics as was the case for the two body problem. To be sure the initial position and velocities of the components would provide the l8 constants required for the unique solution of the motion since the laws of Newtonian mechanics are deterministic. But these initial values are not integrals of the motion. The parameters they specify are not constant during the motion of the members of the system. Thus while they provide a basis for calculating the motion of a specific system, they do not allow for a general solution. Since the general solution of the three body problem appears beyond our grasp, let us consider a simpler problem intermediate between the two body problem and the general three body problem.

The question of what is the most complicated problem in celestial mechanics that allows for a general solution has occupied some of the best analytical minds of the past three centuries and continues to be of interest today. Consider two bodies of comparable but dissimilar masses in circular orbit about one another. Now introduce a third object of negligible mass. Here "negligible mass" means that it is affected by the presence of the other two objects, but does not exert sufficient force on either of the two so as to disturb their circular motion. It is then a reasonable question to inquire into the motion of this third object. Such a question is not entirely academic as this is an excellent approximation to the motion of a spacecraft in the earth-moon system. It is also a fair approximation to the motion of some asteroids influenced by the gravitational fields of the sun and Jupiter. This problem is called the circular restricted three body problem and its solution contains some surprising results.

a. Jacobi's Integral of the Motion

We analyzed the two body problem in physical units, but we are free to choose any system of units we please. So let us measure mass in units such that the total mass of the system is unity. Then

$$m_1 + m_2 = M \equiv 1 \quad . \tag{8.1.1}$$

We could then quite arbitrarily require the less massive of the orbiting pair. to have a mass μ so that

$$\left.\begin{array}{l} m_1 \equiv \mu \\ m_2 = 1-\mu \\ \mu \le \frac{1}{2} \end{array}\right\} \quad . \tag{8.1.2}$$

Remember that the third object in the system has essentially zero mass in that it doesn't contribute to the total mass of the system at any level that could be considered significant. Indeed, it behaves as a 'test particle' as described in Chapter 5. Now we are free to choose the units by which we measure time so instead of using seconds, let us measure time in units of the orbital period of the two significant objects about one another. For the earth and the sun this would be years multiplied by 2π. Such a choice requires that the attractive force between the objects be such that

$$\omega \equiv k\{[(1-\mu)+\mu]/d^3\}^{1/2} = 1 \quad . \tag{8.1.3}$$

Now for the description of the motion of the third object, let us choose a Cartesian coordinate system with an origin at the center of mass and rotating with the uniform circular motion of the two non-negligible masses. Thus the least massive object will be located at x_1 and the more massive one at x_2. The third object will have coordinates [x,y,z] so that its radial distance from the two objects can be represented by

$$\left.\begin{aligned} r_1^2 &= (x-x_1)^2+y^2+z^2 \\ r_2^2 &= (x-x_2)^2+y^2+z^2 \end{aligned}\right\} \quad . \tag{8.1.4}$$

In an inertial coordinate system the total energy would simply be

$$\tfrac{1}{2}\sum_i m_i v_i^2 - (Gm_1m_2)/d = E \quad . \tag{8.1.5}$$

However, if the coordinate system is rotating, the kinetic energy will be reduced by the rotational motion and, to conserve energy, we will have to increase the potential energy by a corresponding amount. Since the orbits of m_l and m_2 are circular, their contribution to the kinetic and potential energies of the system will separately remain constant. Thus energy conservation can be reduced to the energy of the small mass body constant. If we let the object have a mass ε, then the total energy of the small body is

$$\tfrac{1}{2}\varepsilon v^2 - \tfrac{1}{2}\varepsilon\omega^2(x^2+y^2) - \frac{\varepsilon(1-\mu)}{r_1} - \frac{\varepsilon\mu}{r_2} = \text{const.} = E(\varepsilon) \quad . \tag{8.1.6}$$

Here v is the velocity measured in the rotating coordinate frame and the quantity $\frac{1}{2}\varepsilon\omega^2(x^2+y^2)$ is just the contribution from the rotational motion of the coordinate frame itself. Dividing out the negligible mass of the third body and taking ω = l, we can write

$$v^2 = \omega^2(x^2+y^2) + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C \quad , \tag{8.1.7}$$

where C is some constant of the motion. This is known as *Jacobi's integral* and is nothing more than the energy integral for the third body. Now it is clear why the orbits of the other two bodies were assumed to be circular. Still the equations of motion for the third object require six constants of motion for complete specification of the motion of the third body. Thus we need five more. The total angular momentum of the system is conserved, but it is entirely tied up in the motion of the two objects and thus is of little help here. The remaining five constants are simply not known, so that it is remarkable that we may say anything about the motion of the third object.

b. Zero Velocity Surfaces

Now $v^2 \geq 0$ by definition so that Jacobi's integral places limits on where the third object may go depending on the value of C. Let us consider surfaces where v = 0, These are surfaces that provide bounds for the third object's motion, for the particle cannot cross them. For it to do so the square of the particle's velocity would have to change sign, Remembering that we can take ω = 1, we can write the expression for the zero velocity surfaces as

$$(x^2 + y^2) + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} = C \quad . \tag{8.1.8}$$

Clearly the value of C must always be positive. Therefore consider the case where C >> 0. Then either $C \approx (x^2 + y^2)$ or one of the radial distances [r_l , r_2] must be small. Thus either the third body is very close to one of the objects in a tight orbit about it or it is very distant and moves as if the pair was a point source. This is the solution most commonly found in nature, The zero-velocity surfaces would then consist of a cylinder normal to the x-y plane at some distance rather greater than the separation distance d and two smaller 'egg- shaped' surfaces close to each of the objects. These surfaces confine the motion to outside the cylinder or within the oval surfaces. As the value of C is decreased, the outer cylinder decreases in radius and the inner ovoids become bigger. As the value of C continues to decrease the two inner ovoids will touch at a point along the line joining two circularly revolving objects. Let this value be called C_2 and the physical point in space labeled L_l. A particle confined within the ovoids will then be able to move from one to another as this "double point" no longer divides regions of space where v^2 has opposite sign, As the ovoids continue to grow with decreasing C they join at L_l forming a hour glass shaped structure that grows to meet the shrinking cylinder. Eventually, as C takes on smaller and smaller values, the two regions will meet first along the line joining the centers and behind the less massive of the two principle masses. Let this value of C be called C_3 and the corresponding spatial location be known as L_2. The point that occurs behind the more massive of the two objects is known as L_3 and occurs when C decreases to C_4. A further decrease in the value of C causes the surfaces to separate into two comma shaped regions in the x-y plane which asymptotically approach two points when C becomes C_5. These two points can be distinguished in that one leads the more massive object in its orbit while the other trails behind. They are called L_4 and L_5 respectively. The L_is are collectively called the Lagrange points and have special significance. Figure 8.1 shows cross sections of these surfaces in the x-y and x-z planes.

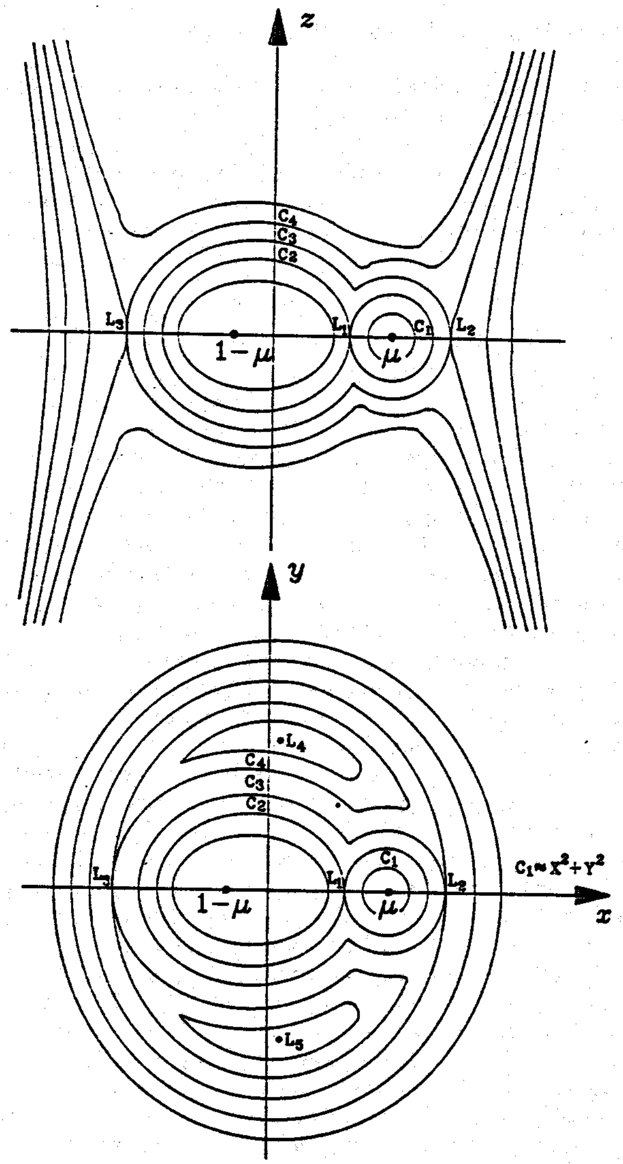

Figure 8.1 shows the zero velocity surfaces for sections through the rotating coordinate system. The upper drawing shows the cross section through the x-z plane while the lower drawing shows the cross section of the x-y plane. The various values of C, as well as the location of the Lagrangian points of equilibrium, are indicated.

c. The Lagrange Points and Equilibrium.

In Chapter 5 [equation (5.3.4)] we defined a "rotational potential" to account for the centrifugal forces generated by the conservation of angular momentum. In a similar manner, we can define a new potential to take account of the rotation of the coordinate frame by including the energy resulting from the motion of the coordinate frame itself. Let this potential be

$$\varnothing = -\tfrac{1}{2}\omega^2(x^2 + y^2) - (1-\mu)/r_1 - \mu/r_2 \quad , \tag{8.1.9}$$

so that the total energy of the third body is

$$E(\varepsilon) = \tfrac{1}{2}\varepsilon v^2 + \varepsilon\varnothing \quad . \tag{8.1.10}$$

The forces acting on the third body will just be

$$\vec{F}(\varepsilon) = -\nabla\varnothing = -\nabla\left(\varnothing - \tfrac{1}{2}C\right) = -\left(\tfrac{1}{2}\right)\nabla\left(v^2\right) \quad . \tag{8.1.11}$$

Since the function v^2 vanishes at the points where two zero velocity surfaces meet and $v^2 \geq 0$, its gradient must also vanish on those surfaces. Thus the points of tangency represent places of equilibrium where all forces on the third body vanish. It remains to be established if those points represent stable equilibrium. Therefore the Lagrangian points may be found from

$$\tfrac{1}{2}\nabla^2\left(v^2\right) = 0 = \hat{i}v\frac{\partial v}{\partial x} + \hat{j}v\frac{\partial v}{\partial y} + \hat{k}v\frac{\partial v}{\partial z} \quad . \tag{8.1.12}$$

Since each component of the vector must be zero separately, the equations of condition for the Lagrangian points are

$$\left.\begin{aligned} v\frac{\partial v}{\partial x} &= x - \frac{(1-\mu)(x-x_1)}{r_1^3} - \frac{\mu(x-x_2)}{r_2^3} = 0 \\ v\frac{\partial v}{\partial y} &= y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} = 0 \\ v\frac{\partial v}{\partial z} &= z\left[\frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3}\right] = 0 \end{aligned}\right\} \quad . \tag{8.1.13}$$

Neither r_1 nor r_2 is zero and $\mu \neq 1$ so that the z-component of the gradient requires that z = 0 and that all the Lagrangian points lie in the x-y plane. If $y \neq 0$, then the y-component of the gradient requires that

$$1-(1-\mu)/r_1^2-\mu/r_2^2=0 \quad . \tag{8.1.14}$$

This has a solution for

$$r_1=r_2=1 \quad . \tag{8.1.15}$$

Since in the units we are using the separation of the two orbiting masses is unity, these points must lie at the vertices of equilateral triangles in the x-y plane having the line joining the two orbiting masses as a base. These are the points L_4 and L_5. Thus the Lagrangian points L_4 and L_5 lie in the orbital plane, leading and following the orbiting bodies by 60° at a distance equal to the separation of those two bodies. If we satisfy the conditions on the gradient by requiring both y and z to be zero, then the x-component of the gradient requires

$$x-\frac{(1-\mu)(x-x_1)}{|x-x_1|^3}-\frac{\mu(x-x_2)}{|x-x_2|^3}=0 \quad , \tag{8.1.16}$$

and the remaining Lagrangian points will lie along the x-axis at the roots of the polynomic equation (8.1.16). In general, all solutions must be found numerically. However, Moulton[8] (p.290) gives series solutions for the location of the Lagrangian points in terms of μ.

In order to test the nature of the stability of the Lagrangian points one need calculate

$$\frac{\partial^2\varnothing}{\partial x_i^2}<0 \quad . \tag{8.1.17}$$

If this condition is satisfied for all of the coordinates, then each of the points represents a point of stable equilibrium. That is, if a particle is slightly displaced from the point, the particle will return to it. This is the case for L_4 and L_5. However, L_l, L_2, and L_3 are all unstable and an object displaced from anyone of them will continue to move away from them. Since the condition given in equation (8.1.17) is essentially the derivative of the forces acting on the particle, stable equilibrium requires that a small displacement generate a small negative force pushing the object back where it came from. A small positive force would continue to accelerate the particle away from its earlier location. The relative stability of the Lagrangian points can be seen from Figure 8.1. For L_l, L_2, and L_3 the touching of the zero velocity surfaces joins two regions where the motion of the particle was previously confined. Thus particles can freely roam from one region to the other. A particle at one of these points could then move either way and would not be stable. However, Lagrangian points L_4 and L_5 represent the 'center' of a forbidden region where $v^2 < 0$ so that the kinetic energy would have to be increased in order to move away from them. As the value of C is decreased so that the forbidden region shrinks to a point, that point can be occupied, but only by a particle with zero velocity. A small displacement would not provide the kinetic energy required for the particle to return to the point and the point would be stable.

The Lagrangian points are important in astronomy as they mark places where particles can either be trapped (L_4 and L_5) or will pass through with a minimum expenditure of energy. In the solar system there are two sets of asteroids known as the Trojan asteroids that lead and follow Jupiter about in its orbit oscillating about L_4 and L_5. In the theory of binary star evolution, the more massive component will expand as it ages until material meets one of the Lagrangian points. If that point is L_l, the matter will stream across the gap between the two stars and eventually be accreted onto the other member of the system. Should either L_2 or L_3 be encountered, the matter will pass through and is likely to be lost to the system entirely.

Much more could be said (eg. Moulton[8]) about the restricted three body problem as books have been written on the subject and some people have devoted their lives to its study. However, its most important aspects are bound up in the study of Jacobi's Integral and it is remarkable that so much can be said about the motion of the third body from knowledge of one integral of the motion.

8.2 The N-Body Problem

After encountering the difficulties posed by the three body problem it must seem foolhardy to even consider larger systems. However, the universe is full of systems of many objects that are largely bound by their mutual gravity and we would like to understand as much about their dynamics as possible. Let us begin by determining the equations of motion for such a system. We can do this as we did for central forces and the two body problem by calculating the Lagrangian. Thus,

$$\mathscr{L} = \sum \tfrac{1}{2} m_i (\dot{\vec{r}}_i \bullet \dot{\vec{r}}_i) + \tfrac{1}{2} \sum_{i \neq j} \sum_j \frac{G m_i m_j}{d_{ij}} \quad . \tag{8.2.1}$$

The equations of motion are therefore

$$m_i \ddot{\vec{r}}_i = \tfrac{1}{2} G m_i \left[\sum_{j \neq i} \frac{m_j (\vec{r}_i - \vec{r}_j)}{d_{ij}^3} \right] \quad . \tag{8.2.2}$$

Summing these equations over all the particles we get

$$\sum_i m_i \ddot{\vec{r}}_i = G \sum_i \sum_{j \neq i} \frac{m_i m_j (\vec{r}_i - \vec{r}_j)}{d_{ij}^3} \quad . \tag{8.2.3}$$

Now since

$$\vec{r}_i - \vec{r}_j = -(\vec{r}_j - \vec{r}_i) \quad , \tag{8.2.4}$$

we may pair the terms in the double sum on the right hand side of equation (8.2.3) so that they individually cancel to zero leaving

$$\sum_i m_i \ddot{\vec{r}}_i = 0 \quad . \tag{8.2.5}$$

This equation can be directly integrated twice with respect to time to get

$$\sum_i m_i \vec{r}_i = \vec{A}t + \vec{B} \quad . \tag{8.6.7}$$

The left hand side of this equation is the definition of the center of mass and the vectors on the right hand side have six linearly independent components. Thus, even for a dynamical system of N particles, the center of mass will move with a uniform velocity. However, N second order vector equations will require 6N constants of integration in order to uniquely specify the motion of the particles and finding six seems of little help.

Taking the cross product of a position vector with the equations of motion we can write

$$\vec{r}_i \times \dot{\vec{r}}_i = G m_i \sum_{j \neq i} m_j \left(\vec{r}_i \times \vec{r}_i - \vec{r}_i \times \vec{r}_j \right) / d_{ij}^3 = G \sum_{j \neq i} \left(\vec{r}_i \times \vec{r}_j \right) / d_{ij}^3 \quad . \tag{8.2.7}$$

Since

$$\vec{r}_i \times \vec{r}_j = -(\vec{r}_j \times \vec{r}_i) \quad , \tag{8.2.8}$$

we can again sum equation (8.2.7) over all the particles and pair the terms under the double sum of the right hand side so that they vanish to zero. Thus we may write

$$\sum_i \vec{r}_i \times m_i \ddot{\vec{r}}_i = \sum_i \vec{r}_i \times \dot{\vec{p}}_i = \frac{d}{dt} \sum_i \left(\vec{r}_i \times \vec{p}_i \right) = \frac{d}{dt} \sum \vec{L}_i = \frac{d\vec{L}}{dt} = 0 \quad , \tag{8.2.9}$$

and find that the total angular momentum of all the particles will be constant. Thus we add three more constants of the motion to our total. We also establish that there will be a fundamental plane of the system that is perpendicular to the total angular momentum vector. Similarly we can invoke the conservation of the total energy to get a last constant of the motion as

$$\tfrac{1}{2}\sum_i m_i \dot{r}_i^2 - \tfrac{1}{2}\sum_j \sum_{i\neq j} G m_i m_j / d_{ij} = E = \text{const.} \tag{8.2.10}$$

Thus, as was the case with the three body problem, we have 10 integrals of the motion, far short of the 6N needed to complete the solution. In addition, all of these constants of the motion are global. That is, they refer to properties of the total system and tell us little about the motion of individual particles. However, there is one more global condition that is of considerable help in understanding the history of the system.

a. The Virial Theorem

The virial theorem, as it is commonly called in the literature, takes on many forms. However, all of them have in common a relationship whose origins are in the equations of motion for the system. We will generate only the simplest of these relationships, namely that appropriate for particles moving under the influence of the gravitational force. A derivation for an arbitrary central force law is given by Collins[9]. The general equations of motion for such a system of particles are given in equation (8.2.2). Now take the scalar product of those equations with a position vector to each object in the system and sum over all the particles so that

$$\sum_i m_i \ddot{\vec{r}}_i \bullet \vec{r}_i = \sum_i m_i \frac{d}{dt}\left(\dot{\vec{r}}_i \bullet \vec{r}_i\right) - \sum_i m_i \dot{\vec{r}}_i \bullet \dot{\vec{r}}_i = G\sum_i m_i \sum_{j\neq i} m_j \frac{\left(\vec{r}_i - \vec{r}_j\right)\bullet \vec{r}_i}{d_{ij}^3} \ . \tag{8.2.11}$$

We can rewrite the central part of equation (8.2.11) so that

$$\tfrac{1}{2}\frac{d^2}{dt^2}\left(\sum_i m_i r_i^2\right) - 2T = G\sum_i\left[\sum_{j>i} m_i m_j\left(\frac{(\vec{r}_i - \vec{r}_j)\bullet \vec{r}_i + (\vec{r}_j - \vec{r}_i)\bullet \vec{r}_j}{d_{ij}^3}\right)\right] \ , \tag{8.2.12}$$

where T is the total kinetic energy of the system. We have also rewritten the left hand side of equation (8.2.11) to explicitly show the pairing of terms for the force of ith particle on the jth particle with the force of the jth particle on the ith particle. The first term in square brackets is a "moment of inertia"-like term only instead of it being a moment of inertia about an axis it is the moment of inertia about the origin of the coordinate system. Let us call this quantity I so that equation (8.2.12) becomes

$$\left.\begin{aligned}\tfrac{1}{2}\frac{d^2I}{dt^2}-2T &= G\sum_i\left[\sum_{j>i} m_i m_j\left(\frac{(\vec{r}_i-\vec{r}_j)\bullet\vec{r}_i+(\vec{r}_j-\vec{r}_i)\bullet\vec{r}_j}{d_{ij}^3}\right)\right] \\ &= G\sum_i\sum_{j>i}\frac{m_i m_j}{d_{ij}}\end{aligned}\right\} \quad . \qquad (8.2.13)$$

The term on the far right is the negative of the potential energy of the system so that

$$\tfrac{1}{2}\frac{d^2I}{dt^2} = 2T - V \quad . \qquad (8.2.14)$$

Some call this the virial theorem, but it is more correctly known as Lagrange's identity even though Lagrange only proved it for the case of three bodies. Karl Jacobi generalized it to a system with N-bodies and it is clearly an identity. That is, it is very like a conservation law as it must always be true for any dynamical system. Now if one integrates Lagrange's identity over time, one can write for stable or bound systems that

$$\frac{\text{Lim}}{t\rightarrow\infty}\left(\frac{1}{t}\int\tfrac{1}{2}\frac{d^2I}{dt^2}dt\right) = 0 = \langle 2T\rangle - \langle V\rangle \quad . \qquad (8.2.15)$$

This result also holds if the system is periodic and the integral is taken over the period of the system, since the system must return to an earlier state so that the moment of inertia and all its derivatives are the same at the limits of the integral. Equation (8.2.15) is known as the time-averaged form of the virial theorem (or generally just the virial theorem) and provides an additional constraint on the behavior of the system.

b. The Ergodic Theorem

The ergodic theorem is in the category of concepts that are so basic that they are never taught, but are assumed to be known. For that reason alone, it is worth discussing. Ergodic theory constitutes a major branch of mathematics and its physical application has occupied some of the best minds of the twentieth century. The theorem from which this branch of mathematics takes its name basically says that the average of some property of a system over all *allowed* points in phase space is identical to the average of that same quantity over the entire lifetime of the system. To explain this and its implications, we must first say what is meant by phase space.

Consider a 6-dimensional space where the coordinates are defined as the location and momentum of a particle. The coordinates of the particle in such a space specify its position and momentum, which requires the six components of its location in phase space. These six components constitute the six constants required for the solution of the Newtonian equations of motion. Thus locating a particle in phase space fixes its entire history - past and future and thus determines the path that the particle will take through phase space as it moves. However, not all points in phase space are allowed to the particle, for its total energy cannot change and there are points in phase space that correspond to different total energies for the particle. Thus, the path of the particle in phase space will be limited to a "space" of one lower dimension - namely one where the total energy is constant. Quantities that limit the phase space available to the motion of a particle are said to be *isolating integrals of the motion* and certainly the total energy is one of them. If we are dealing with the' motion of a single particle in an arbitrary conservative force field, its angular momentum will also be an isolating integral.

Thus, the ergodic theorem says that a particle will reach every point in the phase space allowed to it during its lifetime so that the path of the particle will completely cover the space. Therefore averages of quantities taken over the space are equivalent to averages taken over the lifetime. This seemingly esoteric theorem is of fundamental importance to physics. In thermodynamics we make predictions about time averages of systems but can observe only phase space averages. Thus to relate the two, it is necessary to invoke the "Ergodic hypothesis" -namely, that the ergodic theorem applies to thermodynamic systems. The best justification for this hypothesis is that thermodynamics works!

Unfortunately, the ergodic theorem has never been proved in its full generality, but sufficiently general versions of it have been proved so that we may use it in science. This allows us to replace the time averages that appear in equation (8.2.15) with averages over phase space. This is fortunate as the average astronomer doesn't live long enough to carry out the time averages required to use the virial theorem. The difficulty in applying the virial theorem is in deciding exactly in what subspace is the system ergodic; that is, in deciding how many isolating integrals of the motion there are and what are they. Without that information, we cannot determine how to carry out the averages over the appropriate phase space.

What sorts of things might we want to average? Clearly for the virial theorem we would like to know the average of the kinetic and potential energies for if they do not satisfy equation (8.2.15), the system is not stable and will eventually disperse. Conversely, if the system is adjudged to be a stable system, the average of one of these quantities, together with the virial theorem, will provide the other. This is often used to determine the mass of stable systems.

c. Liouville's Theorem

We conclude our discussion of the N-body problem with a brief discussion of a theorem that deals with the history of an entire system of particles. To do this, we need to generalize our notion of phase space. Consider a space of not just six dimensions, but 6N dimensions where N is the number of particles in the system. Each of the dimensions represents either the position or momentum of one of the particles. As there was need of six dimensions for a system consisting of one particle, the 6N dimensions will suffice to specify the initial conditions for every particle in the system. Thus, the system represents only a point in this huge space, and the space itself is the space of all possible systems of N particles. Such a space is usually distinguished from phase space by calling it configuration space. The temporal history of such a system will be but a single line in configuration space. Liouville's theorem states that the density of points in configuration space is constant. This, in turn, can be used to demonstrate the determinism and uniqueness of Newtonian mechanics. If the configuration density is constant, it is impossible for two different system paths to cross, for to do so, one path would have to cross a volume element surrounding a point on the other path thereby changing the density. If no two paths can cross, then it is impossible for any two ensembles ever to have exactly the same values of position and momentum for all of their particles. Equivalently, the initial conditions of an ensemble of particles uniquely specify its path in configuration space. This is not offered as a rigorous proof, but only as a plausibility argument. More rigorous proofs can be found in any good book on classical mechanics.

The ergodic theorem applies here as well, for if any two systems ever cross in configuration space, they must in reality be the same system seen at different times in its dynamical history. Clearly the paths of systems with different total energies can never cross in accord with Liouville's theorem, but will cover the subspace allowed to them in accordance with the ergodic theorem.

These three theorems are powerful products of the great development of classical mechanics of the nineteenth century. They give us additional and rigorous constraints that apply to systems with any number of particles and they lie at the very foundations of modern physics. They are basically statements of conservation laws and the determinism of Newtonian physics.

8.3 Chaotic Dynamics in Celestial Mechanics

Theoretical physics has had a difficult time, in general, describing phenomena that exhibit some degree of order, but not complete order. Total disorder can be dealt with and thermodynamics is an example of highly developed theoretical structure that deals with gases whose constituents show totally random behavior. Classical mechanics describes well ordered systems with great success. However, intermediate cases are not well understood. This weakness in theoretical physics can be found throughout the discipline from the theories of radiative and convective transfer of energy, to "cooperative phenomena" in stellar dynamics. We have seen from our study of the N-body problem that non-periodic solutions and ergodic paths in phase space can result. The solar system is an N-body system, yet it clearly displays a high degree of order. Might not we expect some aspects of it to behave otherwise?

The space program of the 1960s and 1970s brought us detailed photographs of various objects in the solar system whose dynamical behavior proved to be far more complicated than was previously imagined. The rings of Saturn proved to be more numerous and structured than anyone believed possible. One of the Saturnian satellites (Hyperion) appears to tumble in an unpredictable manner. The rings of Uranus have a structure that most astronomers would have thought was unstable. This list is far from exhaustive, but begins to illustrate that there are many problems of celestial mechanics that remain to be solved. One of the most productive approaches to some of these problems has been through the developing mathematics of Chaos. In the area of dynamics, chaotic phenomena are those that, while being restricted in phase space, do not exhibit any discernable periodicity. Wisdom[10] has written a superb review article on the examples of chaos in the dynamics of the solar system and we will review some of his observations.

In the nineteenth century Poincare showed that integrals of the motion usually do not survive orbital perturbations. Thus, closed form integration of perturbed orbits will not, in general, be possible. However, a more recently proved theorem known as the KAM theorem shows that for small perturbations, orbital motion will remain quasi-periodic. Thus the simple loss of the integrals of the motion does not imply that the dynamical motion of the object will become unrestrained in phase-space and be ergodic. This somewhat surprising result implies that we might expect to find orbits that are largely unpredictable but remain confined to parts of phase space. Wisdom points out that the phase space accessible to a system with a given Hamiltonian may depend critically on the initial conditions. For some sets of initial conditions, the motion of the system will be quasi-periodic, and the system will be confined to a relatively small volume of phase space. For modest changes in the initial conditions, the motion of the system becomes chaotic and completely unpredictable. It is a characteristic of such systems that the transition from one region to another is quite sharp. A similar situation is seen in thermodynamics when a system undergoes a phase transition. Here the mathematics of Chaos has been relatively successful in describing such transitions.

A simple example of such a dynamical system can be found in the restricted three body problem. From Figure 8.1 it is clear that an object orbiting close to one or the other of the two main bodies will experience nearly elliptical motion that is certainly quasi-periodic. However, for values of C of the order of C3 the motion is barely confined and numerical experiments show that the orbits wander over a large range of the allowable phase space in a non-periodic manner. Thus with chaotic behavior being present in such a relatively simple system, we should not be surprised to find it in the solar system. While analysis of such systems is still in its infancy, we know enough about the mathematics of Chaos to be confident that it will lead to a more complete understanding of non- linear dynamical systems. We are once again reminded that the future of theoretical physics can be seen "through a glass darkly" in the developing mathematics of the present.

Chapter 8: Exercises

1. Show that the Lagrangian points L_4 and L_5 are points of stable equilibrium while the Lagrangian points L_1-L_3 are not.

2. Derive the virial theorem for an attractive potential that varies as $1/r^2$.

3. Show that the virial theorem has its normal form even if there are velocity dependent forces present.

4. How many isolating integrals of the motion are there for the case of just two orbiting bodies? What does this mean for the application of the ergodic theorem to the virial theorem?