3

Sources and Sinks of Energy

• • •

We have come to that place in the study of stellar structure where we must be mindful of the flow of energy through the star. After all, stars do shine. So far, we have been able to learn much about the equilibrium structure of a star without considering that it is really a structure in a steady state, rather than one in perfect strict equilibrium. The basic reason that we have been able to ignore the flow of energy through the star is that, during a dynamical time, a very small fraction of the stored energy in the star escapes from the star. Although a star is not, strictly speaking, an equilibrium structure, it comes closer to being one than most any other object in the universe.

However, before delving into the actual movement of energy within the star, we must first identify the sources of that energy as well as the processes which impede its flow. This will also give us the chance to discuss the stores of energy within the star since these certainly represent a potential supply of flowing energy with which to generate the stellar luminosity.

3.1 "Energies" of Stars

One of the great mysteries of the late nineteenth and early twentieth centuries was the source of the energy required to sustain the luminosity of the sun. By then, the defining solar parameters of mass, radius, and luminosity were known with sufficient precision to attempt to relate them. For instance, it was clear that if the sun derived its energy from chemical processes typically yielding less that 10^{12} erg/g, it could shine no longer than about 10,000 years at its current luminosity. It is said that Lord Kelvin, in noting that the liberation of gravitational energy could only keep the sun shining for about 10 million years, found it necessary to reject Charles Darwin's theory of evolution because there would have been insufficient time for natural selection to provide the observed diversity of species.

a Gravitational Energy

It is generally conceded that the sun has shone at roughly its present luminosity for at least the past 2 billion years and has been in existence for nearly 5 billion years. With this in mind, let us begin our study of the sources of stellar energy with an inventory of the stores of energy available to the sun. Perhaps the most obvious source of energy is that suggested by Lord Kelvin, namely gravitation. From the integral theorems of Chapter 2, we may place a limit on the gravitational energy of the sun by remembering that $I_{1,1}(R)$ is related to the total gravitational potential energy. Thus, from equations (2.2.2) and (2.2.5)

$$\Omega \leq -\frac{3}{5}\frac{GM^2}{R} \tag{3.1.1}$$

The right-hand side of the inequality is the gravitational potential energy for a uniform density sphere, which provides a sensible upper limit for the energy. Remember that the gravitational energy is considered negative by convention; a rather larger magnitude of energy may be available for a star that is more centrally concentrated than a uniform- density sphere. We may acquire a better estimate of the gravitational potential energy by using the results for a polytrope. Chandrasekhar[1] obtains the following result, due to Betti and Ritter, for the gravitational potential energy of a polytrope:

$$\Omega = -\frac{3}{5-n}\frac{GM^2}{R} \tag{3.1.2}$$

For a star in convective equilibrium (that is, $n = 3/2$) the factor multiplying GM^2/R becomes 6/7 or nearly unity. Note that for a polytrope of index 5, $\mathbf{\Omega} \rightarrow -\infty$ implying an infinite central concentration of material. This is also one of the polytropes for which there exists an analytic solution and $\xi_1 = \infty$. Thus, one has the picture of a mass point surrounded by a massless envelope of infinite extent. Equation (3.1.2) also tells us that as the polytropic index increases, so does the central concentration.

It is not at all obvious that the total gravitational energy would be available to permit the star to shine. Some energy must be provided in the form of heat, to provide the pressure which supports the star. We may use the Virial theorem [equation (1.2.35)] to estimate how much of the gravitational energy can be utilized by the luminosity. Consider a star with no mass motions, so that the macroscopic kinetic energy **T** in equation (1.2.35) is zero. Let us also assume that the equilibrium state is good enough that we can replace the time averages by the instantaneous values. Then the Virial theorem becomes

$$2\mathbf{U} + \mathbf{\Omega} = 0 \tag{3.1.3}$$

Remember that **U** is the total internal kinetic energy of the gas which includes all motions of the particles making up the gas. Now we know from thermodynamics that not all the internal *kinetic* energy is available to do work, and it is therefore not counted in the internal energy of the gas. The internal kinetic energy density of a differential mass element of the gas is

$$d\mathbf{U} = (3/2)RTdm = (3/2)(C_P - C_V)Tdm \tag{3.1.4}$$

where the relationship of the gas constant R to the specific heats was given in Chapter 2 [equation (2.4.5)]. However, from the definition of specific heats [equation (2.4.4)], the internal heat energy of a differential mass element is

$$dU = C_V Tdm \tag{3.1.5}$$

Eliminating Tdm from equations (3.1.4) and (3.1.5) and integrating the energy densities of the entire star, we get

$$\mathbf{U} = (3/2) \langle \gamma - 1 \rangle U \tag{3.1.6}$$

where **U** is the total internal heat energy or just the total internal energy. The quantity $\langle\gamma - 1\rangle$ is the value of $\gamma - 1$ averaged over the star. For simplicity, let us assume that γ is constant through out the star. Then the Virial theorem becomes

$$3(\gamma\text{-}1)U + \mathbf{\Omega} = 0 \tag{3.1.7}$$

Remembering that the total energy **E** is the sum of the internal energy and the gravitational energy, we can express the Virial theorem in the following ways:

$$U = \frac{-\Omega}{3(\gamma - 1)}$$
$$\mathbf{E} = -(3\gamma - 4)U$$
$$\mathbf{E} = \frac{3\gamma - 4}{3(\gamma - 1)}\Omega \tag{3.1.8}$$

It is clear that for $\gamma > 4/3$ (that is, $n < 3$), the total energy of the star will be negative. This simply says that the star is gravitationally bound and can be in equilibrium. So we can look for the physically reasonable polytropes to have indices less than or equal to 3. The case of $n = 3$ is an interesting one that we shall return to later, for it represents radiation dominated gas. In the limit of complete radiation dominance, the total energy of the configuration will be zero.

b Rotational Energy

While utilizing the Virial theorem to estimate the gravitational energy, we set the mass motions of the star to zero so that the macroscopic kinetic energy **T** was zero. However, stars do rotate, and we should not forget to count the rotational energy in the inventory of energies. We may place a reasonable upper limit on the magnitude of the rotational energy that we can expect by noting that (1) the moment of inertia of the star will always be less than that of a sphere of uniform density and (2) there is a limit to the angular velocity ω_c at which the star can rotate. Thus, for a centrally condensed star

$$\omega^2 \le \frac{8GM}{27R_p^3} \qquad I_z < \tfrac{2}{5}MR^2 \tag{3.1.9}$$

which implies that the rotational energy must be bounded by

$$E_{\mathrm{rot}} = \tfrac{1}{2}I_z\omega^2 < \frac{8}{135}\frac{GM^2}{R} \tag{3.1.10}$$

One may quibble that we have used the angular velocity limit for a centrally condensed star and the moment of inertia for a uniform-density star, but the fact remains that it is extremely difficult for a star to have more than about 10 percent of the magnitude of its gravitational energy stored in the form of rotational energy.

Table 3.1 Mass Defect for Common Nuclear Fuels

Element	Atomic Weight (relative to ^{12}C)	Mass of H Used	Mass Loss by Fusion, %
H	1.00797	1.00797	—
^{4}He	4.0026	4.03188	0.726
^{12}C	12.000	12.0956	0.791
Fe	55.847	56.847	1.062

c Nuclear Energy

Of course, the ultimate upper limit for stored energy is the energy associated with the rest mass itself. It is also the common way of estimating the energy available from nuclear sources. Indeed, that fraction of the rest mass which becomes energy when four hydrogen atoms are converted to one helium atom provides the energy to sustain the solar luminosity. Below is a short table giving the mass loss for a few common elements involved in nuclear fusion processes.

Table 3.2 Possible Sources of Solar Energy

Form of Energy	Amount, erg	$\%M_\odot$	Lifetime for constant $L_\odot$, years
Chemical (10^{12} erg/g)	1.98×10^{45}	10^{-7}	14,000
Rotational ($8\ \Omega/81$)	2.21×10^{47}	10^{-5}	1.4 million
Gravitational [$3GM^2/(5R)$]	2.24×10^{48}	10^{-4}	14 million
Nuclear ($0.0106 M_\odot c^2$)	1.89×10^{52}	1	140 billion
Rest mass energy ($M_\odot c^2$)	1.78×10^{54}	100	1.4×10^{13}

Clearly most of the energy to be gained from nuclear fusion occurs by the conversion of hydrogen to helium and less than one-half of that energy can be obtained by all other fusion processes that carry helium to iron. Nevertheless, .7 percent of Mc^2 is a formidable supply of energy. Table 3.2 is a summary of the energy that one could consider as being available to the sun. All these entries are generous upper limits. For example, the sun rotates at less than .5 percent of its critical velocity, it was never composed of 100 percent hydrogen and will begin to change significantly when a fraction of the core hydrogen is consumed, and not all the gravitational energy could ever be converted to energy for release. In any event, only nuclear processes hold the promise of providing the solar luminosity for the time required to bring about agreement with the age of the solar system as derived from rocks and meteorites. However, the time scales of Table 3.2 are interesting because they provide an estimate of how long the various energy sources could be expected to maintain some sort of equilibrium configuration.

3.2 Time Scales

One of the most useful notions in stellar astrophysics for establishing an intuitive feel for the significance of various physical processes is the time required for those processes to make a significant change in the structure of the star. To enable us to estimate the relative importance of these processes, we shall estimate the time scales for several of them. In Chapter 2 we used the free-fall time of the sun to establish the fact that the sun can be considered to be in hydrostatic equilibrium. The statement was made that this time scale was essentially the same as the dynamical time scale. So let us now turn to estimating the time required for dynamical forces to change a star.

a Dynamical Time Scale

The Virial theorem of Chapter 1 [equation (1.2.34)] provides us with a ready way of estimating the dynamical time scale, for in the form given, it must hold for all $1/r^2$ forces. Consider a star which is not in equilibrium because the internal energy is too low. As it enters the non-equilibrium condition, the star's kinetic energy will also be small. Thus, the Virial theorem would require

$$\frac{d^2I}{dt^2} \approx \Omega \tag{3.2.1}$$

implying a rapid collapse. If we take as an average value for the accelerative change in the moment of inertia

$$\frac{d^2I}{dt^2} \approx -\frac{I}{\tau_d^2} \tag{3.2.2}$$

where t_d is the dynamical time by definition, then we get

$$\tau_d^2 = \frac{-I}{\Omega} = \frac{\frac{2}{5}MR^2}{\frac{3}{5}GM^2/R} \tag{3.2.3}$$

or

$$\tau_d = \left(\frac{\frac{2}{3}R^3}{GM}\right)^{1/2} \tag{3.2.4}$$

Now we compare this to the free-fall time obtained by direct integration of

$$\frac{d^2r}{dt^2} = -\frac{GM(r)}{r^2} \tag{3.2.5}$$

remembering that, since the star is "free-falling", M(r) will always be the mass interior to r. Thus, a surface point will always be affected by the total mass M. With some attention to the boundary conditions [see equations (5.2.12) through (5.2.17)], direct integration yields a free-fall time of

$$\tau_f = \frac{\pi}{2}\left(\frac{R^3}{2GM}\right)^{1/2} = \left(\frac{3\pi}{32G\langle\rho_i\rangle}\right)^{1/2} \tag{3.2.6}$$

which is essentially the same (within about a factor of 1.4) as the dynamical time.

Although we considered a star having zero pressure in order to derive both those time scales, the situation would not be significantly different if some pressure did exist. While a collapse will cause an increase in the pressure, the Virial theorem assures us that the gravitational energy will always exceed the internal energy of the gas unless there is a change in the equation of state resulting in a sudden increase in the internal energy. However, for the interior of the star to adjust to the collapse, it is necessary for information regarding the collapse to be communicated throughout the star. This will be accomplished by pressure waves which travel at the speed of sound. The sound crossing time is

$$\tau_s = \frac{R}{\langle c_s\rangle} = R\left\langle\frac{\gamma P}{\rho}\right\rangle^{-1/2} = R\left\langle\left(\frac{\gamma kT}{\mu m_h}\right)^{-1/2}\right\rangle \tag{3.2.7}$$

For a monatomic gas $\gamma = 5/3$. Hence

$$\langle c_s\rangle = \left(\frac{5k}{3\mu m_h}\right)^{1/2}\langle T^{1/2}\rangle \tag{3.2.8}$$

We may estimate the mean temperature for a uniform density sphere from the integral theorems [equations (2.2.4) and (2.2.7)] and obtain

$$\tau_s = \left(\frac{3R^3}{GM}\right)^{1/2} \tag{3.2.9}$$

Although the sound crossing time is somewhat larger than the free-fall and dynamical time scales, they are all of the same order of magnitude, $\sqrt{(R^3/GM)}$. This is about 27 min for the sun. The similar magnitude for these times is to be expected since they have a common origin in dynamical phenomena. So we have finally justified our statement in Chapter 2 that any departure from hydrostatic equilibrium will be resolved in about 20 min. This short time scale is characteristic of the dynamical time scale; it is generally the shortest of all the time scales of importance in stars.

b Kelvin-Helmholtz (Thermal) Time Scale

Now we turn to some of considerations that led Lord Kelvin to reject the Darwinian theory of evolution. These involve the gravitational heating of the sun. If you imagine the early phases of a star's existence, when the internal temperature is insufficient to ignite nuclear fusion, then you will have the physical picture of a cloud of gas which is slowly contracting and is thereby being heated. Ultimately some of the energy generated by this contraction will be released from the stellar surface in the form of photons. As long as the process is slow compared to the dynamical time scale for the object, the Virial theorem in the form of equation (1.2.35) will hold and $\langle \mathbf{T} \rangle \approx 0$. Thus

$$\tfrac{1}{2}\langle \mathbf{\Omega} \rangle = - \langle \mathbf{U} \rangle \tag{3.2.10}$$

which implies that one-half of the change in the gravitational energy will go into raising the internal kinetic energy of the gas. The other half is available to be radiated away. This was the mechanism that Lord Kelvin proposed was responsible for providing the solar luminosity and he suggested a lifetime for such a mechanism to be simply the time required for the luminosity to result in a loss of energy equal to the present gravitational energy. If we estimate the latter by assuming that the star of interest is of uniform density, then

$$\tau_{\text{K-H}} = -\frac{\Omega}{L} = \frac{\frac{3}{5}GM^2}{RL} \tag{3.2.11}$$

This is known as the *Kelvin-Helmholtz gravitational contraction time*, and it is the same as the lifetime obtained from the gravitational energy given in the previous section. Since the star is simply cooling off and having its internal energy re-supplied by gravitational contraction, some authors refer to this time scale as the thermal time scale. More properly, one could define the thermal time scale t_{th} as the time required for the luminosity to result in an energy loss equal to the internal heat energy, and then one could relate that to the Kelvin-Helmholtz time by means of the Virial theorem. That is,

$$\tau_{\text{th}} = \frac{\langle U \rangle}{L} = -\frac{\langle \Omega \rangle}{\langle 3(\gamma - 1) \rangle L} \approx \frac{\tau_{\text{K-H}}}{\langle 3(\gamma - 1) \rangle} \tag{3.2.12}$$

Thus, we see that the two time scales are of the same order of magnitude differing only by a factor of 2 for a monatomic gas. For the sun, both time scales are of the order of 10^{11} times longer than the dynamical time. In general the thermal time scale is very much longer than the dynamical time scale. The thermal time scale is the time over which thermal instabilities will be resolved, and so they are always less important than dynamical instabilities.

c Nuclear (Evolutionary) Time Scale

In the beginning of this section we estimated the lifetime of the sun which could result from the dissipation of various sources of stored energy. By far the most successful at providing a long life was nuclear energy. The conversion of hydrogen to iron provided for a lifetime of some 140 billion years. However, in practice, when about 10 percent of the hydrogen is converted to helium in stars like the sun, major structural changes will begin to occur and the star will begin to evolve. We can define a time scale for these events in a manner analogous to our other time scales as

$$\tau_n = \frac{K_n M c^2}{L} \tag{3.2.13}$$

where K_n is just the fraction of the rest mass available to a particular nuclear process. While evolutionary changes often occur in one-tenth of the nuclear time scale, some stars show no significant change in less than $0.99t_n$. While in the terminal phases of some stars' lives the nuclear time scale becomes rather shorter than the thermal time scale and conceivably shorter than the dynamical time scale, for the type of stars we will be considering the nuclear time scale is usually very much longer than the other two. Certainly for main sequence stars we may observe that

$$\tau_d \approx \tau_f \approx \tau_s \ll \tau_{KH} \ll \tau_n \tag{3.2.14}$$

It is important to understand that the time scales themselves may change with time. The nuclear time scale will depend on the nature of the available nuclear fuel. However, the time scales do indicate the time interval over which you may regard their respective processes as approximately constant. They are useful, for they are easy to estimate, and they indicate which processes within the star will be important in determining its structure at any given time.

3.3 Generation of Nuclear Energy

We have established that the most important source for energy in the sun results from nuclear processes. Therefore, it is time that we look more closely at the details of those processes with a view of quantifying the dependence of the energy generation rate on the local values of the state variables. During the last 50 years, great strides have been made in understanding the details of nuclear interactions. They have revealed themselves to be remarkably varied and complex. We do not attempt to delve into all these details; rather we sketch those processes of primary importance in determining the structure of the star during the majority of its lifetime. We will leave to others to describe the spectacular nuclear pyrotechnics which occur during the terminal phases of the evolution of massive stars. Indeed, the equilibrium processes that occur in the terminal phases of stellar evolution, giving rise to most of the heavier elements, are beyond the scope of this book. Nor do we attempt to develop a complete, detailed quantum theory of nuclear energy production. Those who thirst after that specific knowledge are referred to the excellent survey by Cox and Giuli[2] and other references at the end of this chapter. Instead, we concentrate on the physical principles which govern the production of energy by nuclear fusion.

a General Properties of the Nucleus

The notion that the atom can be viewed as being composed of a nucleus surrounded by a cloud of electrons which are confined to shells led to a very successful theory of atomic spectra. A very similar picture can be postulated for the nucleus itself, namely, that nucleons are arranged in shells within the nucleus and undergo transitions from one excited state (shell) to another subject to the same sort of selection rules that govern atomic transitions. The origin of the shell structure of any nucleus is that nucleons are fermions and therefore must obey the Pauli Exclusion Principle, just as the atomic electrons do. Thus, only two protons or two neutrons may occupy a specific cell in phase space (protons and neutrons have the same spin as electrons, so each species can have two of its kind in a quantum state characterized by the spatial quantum numbers).

However, the nucleons are much more tightly bound in the nucleus than the electrons in the atom. Whereas the typical ionization energy of an atom can be measured in tens to thousands of electron volts, the typical binding energy of a nucleon in the nucleus is several million electron volts. This large binding energy and the Pauli Exclusion Principle can be used to explain the stability of the neutron in nuclei. Although free neutrons beta-decay to protons (and an electron and an electron antineutrino) with a half-life of about 10 min, neutrons appear to be stable when they are in nuclei. If neutrons did decay, the resulting proton would have to occupy one of the least tightly bound proton shells, which frequently costs more energy than is liberated by the beta decay of the neutron. Thus, unless the neutron decay can provide sufficient energy for the decay products to be ejected from the nucleus, the neutron must remain in the nucleus as a stable entity.

In general, for a nucleus to be stable, its mass must be less than the sum of the masses of any possible combination of its constituents. Thus, Li^5 is not stable, whereas He^4 is. A more detailed explanation of the reasons for the stability or instability of a particular nucleus requires a considerably more detailed discussion of nuclear interactions and nuclear structure than is consistent with the scope of this book. However, note that the instability of mass-5 nuclei posed one of the greatest barriers of the century to the understanding of the evolution of stars. The nuclear evolution beyond mass 5 was finally solved by Fred Hoyle, who showed that the triple-a process, which we consider later, could actually initiate synthesis of all the nuclei heavier than mass 12.

Before we turn to the specifics of nuclear energy production, it is worth saying something about notation. Consider the reaction where a particle a hits a nucleus *X*, producing a nucleus ϒ and other particle(s) b. In other words,

$$a + X \rightarrow Y + b$$

(3.3.1)

Such a reaction can be written X(a,b) ϒ. Usually for such a reaction to happen, it must be exothermic. That is, the rest energy of the initial constituents of the reaction must exceed that of the products.

$$E_n \equiv (m_a + M_X - m_b - M_Y)c^2 > 0$$

(3.3.2)

b The Bohr Picture of Nuclear Reactions

Although quantum mechanics formally describes the transition from the initial to the final state, it is convenient to break down the process and to say that a compound nucleus is formed by the collision and subsequently decays to the reaction products. With this assumption, a reaction can be viewed as consisting of two steps

$$a + X \rightarrow C^* \rightarrow Y + b$$

(3.3.3)

where $\mathbf{C}^*$ is the compound nucleus and the asterisk indicates that it is in an excited state. The compound nucleus can decay by various modes which have these convenient physical interpretations:

$$a + X \rightarrow C^* \rightarrow X + a \qquad \text{elastic scattering}$$
$$a + X \rightarrow C^* \rightarrow X^* + a \qquad \text{inelastic scattering}$$
$$a + X \rightarrow C^* \rightarrow Y + b \qquad \text{particle emission}$$
$$a + X \rightarrow C^* \rightarrow C + \gamma \qquad \text{radiative capture}$$

(3.3.4)

Elastic scattering simply involves a particle "bouncing off" the nucleus in such a manner that the momentum and kinetic energy of both the constituents are conserved. However, inelastic scattering results in the nucleus being left in an excited state at the expense of the kinetic energy of the reactants. Particle emission is the process most often associated with nuclear reactions. The results of the interaction leave both reactants changed. Under certain conditions, the Bohr picture fails for these interactions since they proceed directly to the final state without the formation of a compound nucleus. In radiative capture, the compound nucleus decays from the excited state to a stable state by the emission of a photon.

The validity of this two-stage process, due to Neils Bohr, depends on the lifetime of the compound nucleus $\mathbf{C}^*$. The duration of a nuclear collision can be characterized by the time it takes for the colliding particle to cross the nucleus. For typical nuclear radii and relative collision speeds of, say 0.1c, this is about 10^{-21} s. If the lifetime of the compound nucleus is long compared to this crossing time, you may assume that the nucleons of the compound nucleus have undergone many "collisions" and that the interaction energy has been statistically redistributed among them. In short, the compound nucleus will have reached statistical equilibrium and reside in a well defined state. In some sense, the compound nucleus can be said to exist. This effectively separates the details of the $\mathbf{C}^* \rightarrow \Upsilon + \mathrm{b}$ reaction from those of the $\mathrm{a} + X \rightarrow \mathbf{C}^*$ reaction. One might say that $\mathbf{C}^*$ will have 'forgotten' about its birth.

More properly, the statistical equilibrium state of $\mathbf{C}^*$ is independent of the approach to that state. This was the case in Chapter 1 where we considered the establishment of statistical equilibrium for a variety of gases. It will also be the case when we consider the details of absorption and reemission of photons by atoms much later. Another way of stating this condition is to say that the average distance between collisions with the nucleons (the mean free path) is much less than the size of the nucleus. Experimentally, this appears to be true for collision energies below 50 Mev. Thus, if the energy is shared among more than a half dozen nucleons, any given nucleon will not have sufficient energy to exceed the binding energy and escape. The result is the formation of a stable nucleus by means of radiative capture.

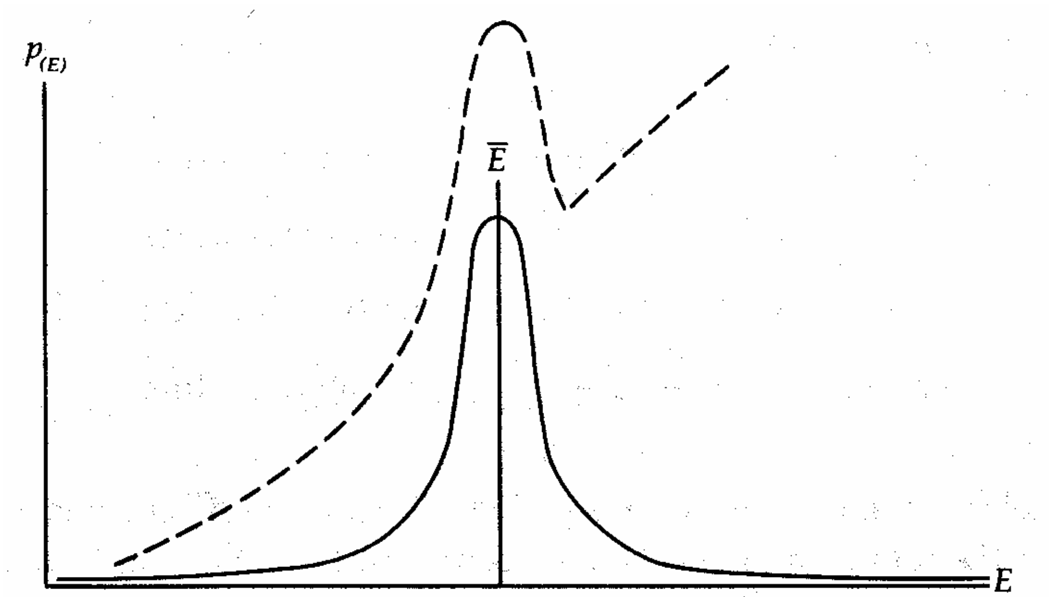

Figure 3.1 shows a typical damping, or dispersion profile. A marked increase in the interaction probability occurs in the vicinity of the resonance energy $\overline{\mathrm{E}}$. The width of the curve is characterized by the damping constant Γ.

By analogy to the photoexcitation of atoms, called *bound-bound transitions*, there exist resonances for nuclear reactions, particularly at low energy. A resonance is an enhancement in the probability that a nuclear reaction will take place. Classically, one may view these as collision energies which excite particular nucleon shell transitions within the nucleus. These energies will be particularly favored for interactions and are known as the *resonance energies*. The probability density distribution with energy is characterized by a function known as a *damping*, or *dispersion*, *profile* whose form we will derive in some detail when we consider the formation of spectral lines in Chapter 13. All that need be understood is the general topological shape (see Figure 3.1) and the fact that the width of the probability maximum can be characterized by a width in energy usually denoted by Γ. As long as the resonance is a simple one and not blended with others, the energy at which the peak of the probability distribution occurs is known as the resonance energy.

c Nuclear Reaction Cross Sections

The words *cross section* have come to have a somewhat generic meaning in nuclear physics as a measure of the likelihood of a particular reaction taking place, in the sense that the larger the cross section, the greater the probability that the reaction will happen. The simplest way to visualize a reaction cross section is to consider the classical notion of a collision cross section. If you were to shoot a bullet through a swarm of hornets, the probability of hitting a particular hornet would be proportional to the cross-sectional area of the hornet as seen by the bullet. Of course, the cross-sectional area of the bullet will also play a role in determining the likelihood of hitting the hornet. The combined effect of these two cross-sectional areas is said to represent the geometric cross section of the collision. In a similar manner, one may interpret a nuclear reaction cross section as the "effective" geometric cross-sectional area of a collision between the particle and the nucleus. Remember that this is not a simple geometric cross section unless you are comfortable with the notion that the nucleus appears to have very different "sizes", as seen by the colliding particle, depending on the particle's energy.

In practice, the nuclear cross section will depend on all the quantities that govern the interaction between the colliding particles and the nucleons in the shell structure of the nucleus. The detailed calculation is usually very complicated, depending on the approximate wave function of the nucleus and the wave function of the colliding particle. A common approximation formula for nuclear cross sections known, as the *Breit-Wigner 1-level dispersion formula*, is

$$\sigma(a,b) = (2l+1)\pi\lambda\!\!\!^{-2}\omega T_l(a)Y(E)SG(b) \tag{3.3.5}$$

where

$$
\begin{aligned}
ƛ &= \frac{\lambda}{2\pi} = \frac{\hbar}{p} = \frac{\hbar}{\sqrt{2m_a E}} \\
L &= \hbar[l(l+1)]^{1/2} = \text{orbital angular momentum of } m_a \text{ about } X \\
T_l(a) &= \text{transmission function of particle } a \\
\omega &= \frac{2J+1}{(2I_a+1)(2I_x+1)} \sim 1 \\
I_a &= \text{spin of particle } a \\
I_x &= \text{nuclear spin of particle } X \\
\vec{J} &= \vec{I}_a + \vec{I}_x \\
&\Upsilon(E) \text{ allows for resonances} \\
S &= 1 \text{ or } 2 \text{ depending on particle degeneracy} \\
G(b) &\sim \frac{\Gamma(b)}{\Gamma} = \text{branching ratio for } b \\
&\Gamma = \Gamma_\gamma + \sum_i \Gamma_i \\
\Gamma_\gamma &= \text{damping constant for radiative capture (i.e., particle } b \text{ is a photon)} \\
\Gamma_i &= \text{all other possible decay damping constants}
\end{aligned}
\tag{3.3.6}
$$

We will make no attempt to derive this result. However, we do try to show that the result at least contains the right sort of terms and is reasonable. The term $\pi ƛ^2$ is essentially the geometric cross section of the colliding particle as it is related to the particle's de Broglie wavelength. The angular momentum term $(2\ell+1)$ is a measure of the impact parameter and the energy. As ℓ increases, so does the impact parameter. For constant angular momentum, an increasing impact parameter will mean a decreasing collision energy, implying a net increase of the collision probability. However, as the impact parameter increases and the collision energy drops, the probability that the colliding particle will be able to overcome the coulomb barrier decreases drastically. Thus, we need be concerned only with $\ell = 0$, or 1. The term *transmission function* of particle a includes the probability that the particle will penetrate the coulomb barrier of the nucleus. The parameter ω allows for the spin-spin interactions of the nucleus and the particle and is of the order unity. Function ϒ(E) includes the effects of resonances and from the dispersion curve in Figure 3.1 can clearly be a very strong function of collision energy E. The spin degeneracy parameter $\boldsymbol{S}$ is generally 1 except when a and X are the same kind of particle and also have zero spin; then $\boldsymbol{S} = 2$. Finally, G(b) is a measure of the probability that particle b will be created from the compound nucleus as opposed to some other possibility. Now that we have the nuclear reaction cross sections, we have to determine the rate at which collisions will occur. Then we will be able to find the energy produced by stellar material.

d Nuclear Reaction Rates

The reaction cross section of the previous section can be measured as a function of the collision energy (and some atomic constants) alone and therefore can be written as a function of the particle's velocity v relative to the target. By resurrecting the geometric interpretation of the cross section, the number of particles crossing an area (colliding with the target) per unit time is just $N\sigma(v)v$ where N is the density of colliding particles (see Figure 3.2)

Consider collisions between two different kinds of particles with a number density in phase space of dN_1 and dN_2. To obtain the number of collisions per second per unit volume, we must integrate over all available velocity space. That is, we must sum over the collisions *between* particles so that the collision rate r is

$$r = \iint v\sigma(v)\, dN_1(\vec{v}_1)\, dN_2(\vec{v}_2) \tag{3.3.7}$$

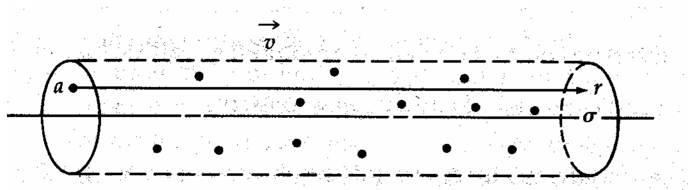

Figure 3.2 is a schematic representation of a collision between particle a and a target with a geometrical cross section σ.

Let us assume that the velocity distributions of both kinds of particles are given by maxwellian velocity distributions

$$dN = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/(2kT)}\, dv \tag{3.3.8}$$

so that equation (3.3.7) becomes

$$r = N_1 N_2 \left(\frac{m_1}{2\pi kT}\right)^{3/2} \left(\frac{m_2}{2\pi kT}\right)^{3/2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\left(\frac{-m_1 v_1^2}{2kT} - \frac{m_2 v_2^2}{2kT}\right) v\sigma(v)\, d\vec{v}_1\, d\vec{v}_2 \tag{3.3.9}$$

If we transform to the center-of-mass coordinate system, assuming the velocity field is isotropic so that the triple integrals of equation (3.3.9) can be written as spherical "velocity volumes", then we can rewrite equation (3.3.9) in terms of the center of mass velocity v_0 and the relative velocity v as

$$r = N_1 N_2 \left(\frac{m_1}{2\pi kT}\right)^{3/2}\left(\frac{m_2}{2\pi kT}\right)^{3/2} \int_0^\infty \int_0^\infty \exp\left[\frac{(m_1+m_2)v_0^2+\tilde{m}v^2}{2kT}\right] \cdot v\sigma(v)4\pi v_0^2(4\pi v^2)\,dv_0\,dv \tag{3.3.10}$$

where

$$\tilde{m} = \frac{m_1 m_2}{m_1+m_2}$$
$$\vec{v} = \vec{v}_1 - \vec{v}_2$$
$$\vec{v}_0 = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} \tag{3.3.11}$$

The integral over v_0 is analytic and is

$$\int_0^\infty e^{-(m_1+m_2)v_0^2/(2kT)} 4\pi v_0^2\,dv_0 = \left(\frac{2\pi kT}{m_1+m_2}\right)^{3/2} \tag{3.3.12}$$

which reduces equation (3.3.10) to

$$r = N_1 N_2 \left(\frac{\tilde{m}}{2\pi kT}\right)^{3/2} \int_0^\infty e^{-\tilde{m}v^2/(2kT)} v\sigma(v)(4\pi v^2)\,dv \tag{3.3.13}$$

Since the *relative* kinetic energy in the center of mass system is $E = \frac{1}{2}\tilde{m}v^2$, we can rewrite equation (3.3.13) in terms of an average reaction cross section $<\sigma(v)\cdot v>$ so that

$$r = N_1 N_2 \langle \sigma(v)v \rangle \tag{3.3.14}$$

where

$$\langle\sigma(v)v\rangle = \frac{2}{\sqrt{\pi}}\left(\frac{1}{kT}\right)^{3/2} \int_0^\infty e^{-E/(kT)}\sigma(v)vE^{1/2}\,dE = \int_0^\infty n(E)v\sigma(v)\,dE \tag{3.3.15}$$

Thus $<\sigma(v)\cdot v>$ is the "relative energy" weighted average of the collision probability of particle 1 with particle 2. When this average cross section is written, the explicit dependence on velocity is usually omitted, so that

$$\langle\sigma v\rangle \equiv \langle\sigma(v)v\rangle \tag{3.3.16}$$

If the collisions involve identical particles, then the number of *distinct* pairs of particles is N(N-1)/2 so the factor of N_1N_2 in equation (3.3.14) is replaced by $N^2/2$.

If we call the energy produced per reaction Q, we can write the energy produced per gram of stellar material as

$$\epsilon = \frac{r_{a,X}Q}{\rho} = \frac{N_a N_X \langle \sigma v \rangle Q}{\rho} \tag{3.3.17}$$

The number densities can be replaced with the more common fractional abundances by mass to get

$$\epsilon = \left[N_0^2 Q \langle \sigma v \rangle \left(\frac{X_X}{m_X} \right) \left(\frac{X_a}{m_a} \right) \right] \rho \tag{3.3.18}$$

where N_0 is Avogadro's number. Since $<\sigma v>$ is a complicated function of temperature and must be obtained numerically, equation (3.3.18) is usually approximated numerically as

$$\epsilon \approx \epsilon_0 \rho T^{\nu} \tag{3.3.19}$$

where

$$\epsilon_0 = \frac{N_0^2 (X_a/m_a)(X_X/m_X)(Q\langle \sigma v \rangle |_{T_0})}{T_0^{\nu}} \tag{3.3.20}$$

Here ν itself is very weakly dependent on the temperature. Most of the important energy production mechanisms have this form. Equation (3.3.19) expresses the energy generated for a specific energy generation mechanism in terms of the state variables T and p. This is what we were after. Formulas such as these, where ε_0 has been determined, will enable us to determine the energy produced throughout the star in terms of the state variables. Before turning to the description of processes which impede the flow of this energy, let us consider a few of the specific nuclear reactions for which we have expressions of the type given by equation (3.3.20.)

e Specific Nuclear Reactions

The nuclear reactions that provide the energy for main sequence stars all revolve on the conversion of hydrogen to helium. However, this is accomplished by a variety of ways. We may divide these ways into two groups. The first is known as the proton-proton cycle (p-p cycle) and it begins with the conversion of two hydrogen atoms to deuterium. Several possibilities occur on the way to the production of ^{4}He. These alternate options are known as P2-P6 cycles. In addition to the proton-proton cycle, a series of nuclear reactions involving carbon, nitrogen, and oxygen also can lead to the conversion of hydrogen to helium with no net change in the abundance of C, N, and O. For this reason, it is known as the CNO cycle. These reactions and their side chains as given by Cox and Giuli[2] are given in Table 3.3

Table 3.3 Common Nuclear Reactions

Step	Q MeV	Proton Cycles
1	1.18	$^{1}H(p, \beta^{+}\nu)^{2}H$ } × 2 (P1)
2	5.49	$^{2}H(p, \gamma)^{3}He$ } × 2 (P1)
3	12.86	$^{3}He(^{3}He, 2p)^{4}He$ (P1)
or		
3	1.59	$^{3}He(\alpha, \gamma)^{7}Be$ (P2 & P3)
4	0.06	$^{7}Be(\beta^{-}, \bar{\nu})^{7}Li \rightarrow {}^{7}Li(p, \alpha)^{4}He$, Q = 17.35 MeV (P2)
or		
4	0.13	$^{7}Be(p, \gamma)^{8}B$ (P3)
5*	10.78	$^{8}B \rightarrow {}^{8}Be + \beta^{+} + \nu$ (P3)
6*	0.09	$^{8}Be \rightarrow 2\,^{4}He$ (P3)
or		
3		$^{3}He(\beta^{-}, \nu)^{3}H$ ← Endothermic by 18 KeV (P4)
4		$^{3}H(p, \gamma)^{4}He$ / $^{3}H(^{3}He, np)^{4}He$ / $^{3}H(^{3}H, 2n)^{4}He$ (P4)

Step	Triple-α Process
1	$2(^{4}He) + (\sim 100$ KeV$) \rightarrow {}^{8}Be^{*}$
2	$^{8}Be^{*}(\alpha, {}^{12}C^{*})$
3	$^{12}C^{*} \rightarrow {}^{12}C + 2\gamma + 7.656$ MeV

Q MeV	CNO Cycles	Step
1.94	$^{12}C(p, \gamma)^{13}N$	1
1.51	$^{13}N \rightarrow {}^{13}C + \beta^{+} + \nu$	2*
7.54	$^{13}C(p, \gamma)^{14}N$	3
7.29	$^{14}N(p, \gamma)^{15}O$	4
1.76	$^{15}O \rightarrow {}^{15}N + \beta^{+} + \nu$	5*
4.96	$^{15}N(p, \alpha)^{12}C$ / $^{15}N(p, \gamma)^{16}O$	6
	$^{16}O(p, \gamma)^{17}F$	7
	$^{17}F \rightarrow {}^{17}O + \beta^{+} + \nu$	8*
	$^{17}O(p, \alpha)^{14}N$ / $^{17}O(p, \gamma)^{18}F$	9
	$^{18}F \rightarrow {}^{18}O + \beta^{+} + \nu$	10*
	$^{18}O(p, \alpha)^{15}N$ / $^{18}O(p, \gamma)^{19}F$	11
	$^{19}F(p, \alpha)^{16}O$	12

* Reactions that occur by spontaneous decay and do not depend on the local values of the state variables.

Besides the steps marked with asterisks, which denote reactions that occur by spontaneous decay and do not depend on local values of the state variables, the steps that are the important contributors to the energy supply have their contribution (their Q value) indicated. The energy of the neutrinos has not been included since they play no role in determining the structure of normal stars. When the p-p cycle dominates on the lower main sequence, most of the energy is produced by means of the P1 cycle. The neutrino produced in the fifth step of the P2 cycle is the high energy neutrino which has been detected, but in unexpectedly low numbers, by the neutrino detection experiment of R. Davis in the Homestake Gold Mine. In general, the relative importance of the P1 cycle relative to P2 and P3 is determined by the helium abundance, since this governs the branching ratio at step 3 in the p-p cycle. If ^{4}He is absent, it will not be possible to make ^{7}Be by capture on ^{3}He.

Virtually all the energy of the CNO cycle is produced by step 6 as the production of ^{12}C from ^{15}N is strongly favored. However, all the higher chains close with only the net production of ^{4}He. The first stage of the P4 cycle is endothermic by 18 keV so unless the density is high enough to produce a Fermi energy of 18 keV, the reaction does not take place. This requires a density of $\rho > 2\times10^4$ g/cm^3 and so will not be important in main sequence stars. Once ^{3}H is produced it can be converted to ^{4}He by a variety of processes given in step 4. The last two are sometimes denoted P5 and P6, respectively, and are rare.

While the so-called triple-α process is not operative in main sequence stars, it does provide a major source of energy during the red-giant phase of stellar evolution. The extreme temperature dependence of the triple-a process plays a crucial role in the formation of low-mass red giants and, we shall spend some time with it later. The ^{8}Be* is unstable and decays in an extremely short time. However, if during its existence it collides with another ^{4}He nucleus, ^{12}C can form, which is stable. The very short lifetime for ^{8}Be* basically accounts for the large temperature dependence since a very high collision frequency is required to make the process productive.

The exponent of the temperature dependence given in equation (3.3.20) and the constant ε_0 both vary slowly with temperature. This dependence, as given by Cox and Giuli[2] (p. 486), is shown in Table 3.4.

Table 3.4 Temperature Dependence of ν and ϵ_0

Proton-Proton			CNO Cycle		Triple-α Process		
T_6	ϵ_0 (*cgs*)	ν	ϵ_0 (*cgs*)	ν	T_8	ϵ_0 (*cgs*)	ν
10	7×10^{-2}	4.60	3×10^{-4}	22.9	0.8	2×10^{-12}	49
20	1	3.54	4.5×10^{2}	18	1.0	4×10^{-8}	41
40	9	2.72	3×10^{7}	14.1	2.0	15	19
80	43	2.08	2×10^{11}	11.1	3.0	6×10^{3}	12
100	—	—	2×10^{12}	10.2	4.0	10^{5}	7.9

The temperature T_6 in Table 3.4 is given in units of 10^6. Thus $T_6 = 1$ is 10^6 K. It is a general property of these types of reaction rates that the temperature dependence "weakens" as the temperature increases. At the same time the efficiency ε_0 increases. In general, the efficiency of the nuclear cycles rate is governed by the *slowest* process taking place. In the case of p-p cycles, this is always the production of deuterium given in step 1. For the CNO cycle, the limiting reaction rate depends on the temperature. At moderate temperatures, the production of ^{15}O (step 4) limits the rate at which the cycle can proceed. However, as the temperature increases, the reaction rates of all the capture processes increase, but the steps involving inverse β decay (particularly step 5), which do not depend on the state variables, do not and therefore limit the reaction rate. So there is an upper limit to the rate at which the CNO cycle can produce energy independent of the conditions which prevail in the star. However, at temperatures approaching a billion degrees, other reaction processes not indicated above will begin to dominate the energy generation and will circumvent even the beta-decay limitation.

We have now determined the various sources of energy that are available to a star so that it can shine. Clearly the only viable source of that energy results from nuclear fusion. The condition for the production of energy by nuclear processes can occur efficiently only under conditions that prevail near the center of the star. From there, energy must be carried to the surface in some manner in order for the star to shine. In the next chapter we investigate how this happens and describe the mechanisms that oppose the flow.

Problems

1. Using existing models or a current model interior program, find the expected solar neutrino flux (i.e., the flux of ^{8}B neutrinos) as a function of solar age from the zero age model to the present.

2. What polytrope(s) would you use to describe the structure of the sun? How well do they match the standard solar model?

3. Consider a gas sphere that undergoes a pressure-free collapse. Let the free-fall time for material at the surface R be t_f. Find the mass distribution for an isothermal sphere and polytropes with indices *n* of 3, and 1.5 at:
   a $t = 0.1t_f$,
   b $t = 0.5t_f$, and
   c $t = 0.8t_f$.

4. Use the Virial theorem to find the fundamental radial pulsation period for a star where the equation of state is $P = K\rho^{\gamma}$. Find the behavior of this period as $\gamma \rightarrow \infty$.

5. Find the mass of a main sequence star for which the energy production by the p-p cycle equals that of the CNO cycle.

References and Supplemental Reading

1. Chandrasekhar, S.: *An Introduction to the Study of Stellar Structure*, Dover, New York, 1957, p. 101, eq.90.

2. Cox, J. P., and Guili, R.T.: *Principles of Stellar Structure*, Gordon & Breach, New York, 1968, Chap. 17, pp. 477-481.

I have provided the bare minimum information regarding nuclear energy generation in this chapter. Further reading should be done in:

Clayton, D. D.: *Principles of Stellar Evolution and Nucleosynthesis*, McGraw- Hill, New York, 1968 Chaps. 4, 5, pp. 283-606.

Rolfs, C., and Rodney, W. S.: *Cauldrons in the Cosmos*, University of Chicago Press, Chicago, 1986.

Rolfs, C., and Trautvetter, H. P.: "Experimental Nuclear Astrophysics" *Ann. Rev. Nucl. Part. Sci.* 28, 1978, pp.115-159.

Bahcall, J. N., Huebner, W. F., Lubia, S. H., Parker, P. D., and Ulrich, R. K.: *Rev. Mod. Phy*. 54, 1982, p. 767.

In addition, a good overview to the way in which the rates of energy generation interface with the equations of stellar structure is found in

Schwarzschild, M.: *The Structure and Evolution of the Stars* Princeton University Press, Princeton N.J., 1958, pp. 73-88.